Probability of Exercise in the Call Price

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One question that is often asked of the Black-Scholes formula is "I know that $\Phi(d_2)$ is the probability of the call ending up in-the-money, but what does $\Phi(d_1)$ represent, in terms of a probability?". The answer is that $\Phi(d_1)$ and $\Phi(d_2)$ are both probabilities of the call ending up in-the-money, but under different measures.

Recall that according to the Black-Scholes model, the stock price is driven by the following SDE

$$dS_t = rSdt + \sigma dW_t^{\mathbb{P}}$$

where $\mathbb{P}$ is the risk neutral measure under which $W_t^{\mathbb{P}}$ is Brownian motion. The Black-Scholes call price with maturity $(T-t)$ and strike K is

$$C = e^{-r(T-t)}E^{\mathbb{P}}\left[(S_T - K)^+\right] = e^{-r(T-t)}E^{\mathbb{P}}\left[(S_T - K)\mathbf{1}_{S_T>K}\right] \tag{1}$$

where $\mathbf{1}_{S_T>K}$ is the indicator function, and where the expectation is taken at time t. This can be written as

$$C = e^{-r(T-t)}E^{\mathbb{P}}\left[S_T\mathbf{1}_{S_T>K}\right] - Ke^{-r(T-t)}E^{\mathbb{P}}\left[\mathbf{1}_{S_T>K}\right] \tag{2}$$

Equation (2) is a general expression for the call price when interest rates are constant. In Equation (2), $E^{\mathbb{P}}\left[\mathbf{1}_{S_T>K}\right] = \mathbb{P}(S_T > K)$ and in the special case of Black-Scholes–namely, when S_T follows a lognormal distribution–it is equal to $\Phi(d_2)$ so it's easy to see that $\Phi(d_2)$ is the probability of the call ending in-the-money. The second expectation is, writing $B_t = e^{rt}$

$$e^{-r(T-t)}E^{\mathbb{P}}\left[S_T\mathbf{1}_{S_T>K}\right] = E^{\mathbb{P}}\left[\frac{B_t}{B_T}S_T\mathbf{1}_{S_T>K}\right] = S_tE^{\mathbb{P}}\left[\frac{B_t/B_T}{S_t/S_T}\mathbf{1}_{S_T>K}\right].$$

Now define the Radon-Nikodym derivative as the ratio of numeraires $\mathbb{Z}_T = \frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{S_t/S_T}{B_t/B_T}$. Then the expectation can be changed from measure $\mathbb{P}$ to a new measure $\mathbb{Q}$ by using the Radon-Nikodym derivative as follows.

$$\begin{aligned}
E^{\mathbb{P}}\left[\frac{B_t/B_T}{S_t/S_T}\mathbf{1}_{S_T>K}\right] &= E^{\mathbb{Q}}\left[\frac{B_t/B_T}{S_t/S_T}\mathbf{1}_{S_T>K}\mathbb{Z}_T\right] \\
&= E^{\mathbb{Q}}\left[\mathbf{1}_{S_T>K}\right] \\
&= \mathbb{Q}(S_T > K).
\end{aligned}$$

This is the probability of exercise, but under measure $\mathbb{Q}$. Hence the call price in Equation (2) can be written

$$C = S_t\mathbb{Q}(S_T > K) - Ke^{-r(T-t)}\mathbb{P}(S_T > K). \tag{3}$$

where

$$\begin{aligned}\mathbb{P}\left(S_T > K\right) &= \text{probability of exercise under the original measure } \mathbb{P} \\ \mathbb{Q}\left(S_T > K\right) &= \text{probability of exercise under the new measure } \mathbb{Q}\end{aligned}$$

In the Black-Scholes model, the call price is

$$C = S_t \Phi\left(d_1\right) - K e^{-r(T-t)} \Phi\left(d_2\right). \tag{4}$$

Comparing Equations (3) and (4), we see that in the Black-Scholes model, $\Phi\left(d_2\right) = \mathbb{P}\left(S_T > K\right)$ and $\Phi\left(d_1\right) = \mathbb{Q}\left(S_T > K\right)$.