Relationship Between Butterfly Spreads and the Risk Neutral Density

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In this Note we show that a portfolio based on a butterfly spread can approximate the risk neutral density (RND), and we establish the result that the second derivative of the call price with respect to strike is the RND.

1 The Portfolio

A butterfly spread is constructed with three call options, each with identical maturity T, but with different strikes. In particular, the butterfly spread is

- Long one option at strike $K - dK$,
- Short two options at strike K,
- Long one option at strike $K + dK$.

The portfolio we consider contains $\left(\frac{1}{dK}\right)^2$ units of a butterfly spread. Its time-t value is therefore

$$V(S_t, K, T, dK) = \frac{C(K - dK) - 2C(K) + C(K + dK)}{dK^2} \tag{1}$$

where $C(K) = C(S_t, K, T)$ is the time-t price of a call with strike K and maturity T.

2 The Portfolio Payoff

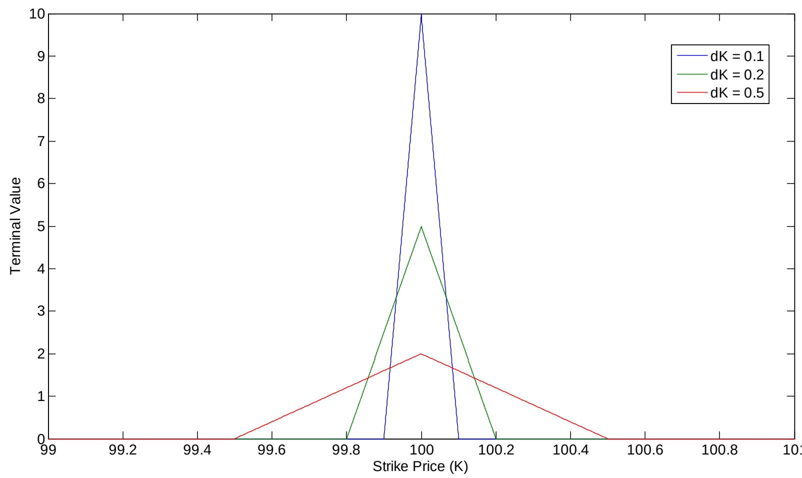
It is straightforward to show that the time-T payoff (terminal value) of the portfolio is

$$V(S_T, K, T, dK) = \begin{cases} \frac{S_T - K + dK}{dK^2} & \text{if } S_T \in [K - dK, K] \\ \frac{K + dK - S_T}{dK^2} & \text{if } S_T \in [K, K + dK] \\ 0 & \text{elsewhere.} \end{cases} \tag{2}$$

which can be written in the more compact form

$$V(S_T, K, T, dK) = \begin{cases} \frac{dK + |S_T - K|}{dK^2} & \text{if } S_T \in [K - dK, K + dK] \\ 0 & \text{elsewhere.} \end{cases} \tag{3}$$

In other words, $V(S_T, K, T, dK) = \frac{dK+|S_T-K|}{dK^2}\mathbf{1}_{S_T\in[K-dK,K+dK]}$. The behavior of the terminal payoff $V(S_T, K, T, dK)$ as dK shrinks is illustrated in the figure below, for three values of dK. It appears that $V(S_T, K, T, dK)$ approaches the Dirac delta function. This will be shown in the next few lines.



Consider the limit of the payoff as the increment dK approaches zero. From Equation (2) we can see that this limit is

$$\lim_{dK\to 0} V(S_T, K, T, dK) = \begin{cases} \infty & \text{if } S_T = K \\ 0 & \text{if } S_T \neq K. \end{cases} \tag{4}$$

The area under $V(S_T, K, T, dK)$ is $\frac{1}{2}\frac{1}{dK}dK + \frac{1}{2}dK\frac{1}{dK} = 1$ for all values of dK. Hence, we can establish a stronger result than Equation (4) that the limit of the terminal payoff of the portfolio is the Dirac delta function

$$\lim_{dK\to 0} V(S_T, K, T, dK) = \delta(S_T - K).$$

3 The Portfolio Value as dK Vanishes

Consider again the time-t value in Equation (1), which can be written as the expected payoff under the risk neutral measure Q, discounted to time t

$$V(S_t, K, T, dK) = P(t, T)E_t^Q\left[V(S_T, K, T, dK)\right]$$

where $P(t,T)$ is the discount factor. Take the limit as $dK \to 0$

$$\begin{aligned}
\lim_{dK\to 0} V(S_t,K,T,dK) &= P(t,T)E_t^Q\left[\lim_{dK\to 0} V(S_T,K,T,dK)\right] \quad (5)\\
&= P(t,T)E_t^Q\left[\delta(S_T-K)\right]\\
&= P(t,T)\int_0^\infty \delta(S_T-K)f_{S_T}(K)dK\\
&= P(t,T)f_{S_T}(K)
\end{aligned}$$

where f_{S_T} denotes the risk neutral density of the terminal stock price S_T. Hence, as the increment dK vanishes, the portfolio approaches the risk neutral density, evaluated at K and discounted back to time t.

4 The Portfolio as a Second-Order Derivative

Note that Equation (1) can be written

$$V(S_t,K,T,dK) = \frac{\left(\frac{C(K-dK)-C(K)}{dK}\right) - \left(\frac{C(K)-C(K+dK)}{dK}\right)}{dK}$$

so that the limit in Equation (5) is also

$$\lim_{dK\to 0} V(S_t,K,T,dK) = \frac{\partial^2 C}{\partial K^2}. \quad (6)$$

Combining Equations (5) and (6) produces

$$P(t,T)f_{S_T}(K) = \frac{\partial^2 C}{\partial K^2},$$

the well-known result that the discounted risk neutral density is the second derivative of the call price with respect to strike.

5 The Butterfly Spread as an Approximation to an Arrow-Debreu Security

As a sidebar, the butterfly spread can also be used to construct a portfolio whose terminal value converges to an Arrow-Debreu security. Consider an alternate portfolio, comprised of $\frac{1}{dK}$ units of a butterfly spread instead of $\left(\frac{1}{dK}\right)^2$ units. This portfolio has the terminal payoff

$$W(S_t,K,T,dK) = \frac{dK + |S_T-K|}{dK}\mathbf{1}_{S_T\in[K-dK,K+dK]}$$

which converges to an Arrow-Debreu security when $dK \to 0$. Indeed, it is clear that the limit as $dK \to 0$ of the payoff is

$$\lim_{dK \to 0} W(S_T, K, T, dK) = \begin{cases} 1 & \text{if } S_T = K \\ 0 & \text{if } S_T \neq K. \end{cases}$$

The behavior of $W(S_T, K, T, dK)$ as dK shrinks is illustrated in the next figure.