1 The SDE and its Transition Density

Start with the SDE defined by
\[ dX_t = \mu(X_t)dt + \sigma(X_t)dW_t. \]

The transition density \( \rho(x, t|y, s) \) is defined by
\[
\rho(x, t|y, s) = \Pr [X_{t+s} \in A|X_s = y] = \Pr [X_t \in A|X_0 = y].
\]

The density \( \rho(x, t|y, s) \) is time-invariant since \( \mu(X_t) \) and \( \sigma(X_t) \) are assumed to be time invariant, and consequently, that \( X_t \) is assumed to be stationary.

2 Derivation of the Equation

Consider a differentiable function \( V(X_t, t) = V(x, t) \) with \( V(X_t, t) = 0 \) for \( t \notin (0, T) \). Then by Itô’s Lemma
\[ dV = \left( \frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} \right) dt + \left[ \sigma \frac{\partial V}{\partial x} \right] dW_t \]
so that
\[
V(X_T, T) - V(X_0, 0) = \int_0^T \left( \frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} \right) dt + \int_0^T \left[ \sigma \frac{\partial V}{\partial x} \right] dW_t \quad (1)
\]
where \( \mu = \mu(X_t) \) and \( \sigma = \sigma(X_t) \) for notational convenience. Take the conditional expectation of both sides of equation (1) given \( X_0 \)
\[
E[V(X_T, T) - V(X_0, 0)] = E\left[ \int_0^T \left( \frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} \right) dt + \int_0^T \left[ \sigma \frac{\partial V}{\partial x} \right] dW_t \right] \quad (2)
\]
In this note, all expectations are expectations conditional on \( X_0 \), so that \( E[\cdot] = E[\cdot|X_0 = y] \). Since \( E[dW_t] = 0 \), the second term in the middle line of equation (2) drops out. Hence, we can write equation (2) as three integrals
\[
\int_{\mathbb{R}} \int_0^T \rho \frac{\partial V}{\partial t} dt dx + \int_{\mathbb{R}} \int_0^T \rho \mu \frac{\partial V}{\partial x} dt dx + \frac{1}{2} \int_{\mathbb{R}} \int_0^T \rho \sigma^2 \frac{\partial^2 V}{\partial x^2} dt dx = I_1 + I_2 + I_3
\]
where $\rho = \rho(x, t | y, s)$ for notational convenience. The objective of the derivation is to apply integration by parts to get rid of the derivatives of $V$.

2.1 Evaluation of the Integrals

The trick is that $I_1$ is evaluated using integration by parts on $t$, while $I_2$ and $I_3$ are each evaluated using integration by parts on $x$.

2.1.1 Evaluation of $I_1$

Use $u = \rho, v' = \frac{\partial V}{\partial t}$ so that $u' = \frac{\partial \rho}{\partial t}$ and $v = V$. Hence for the inside integrand of $I_1$ we have

$$
\int_0^T \rho \frac{\partial V}{\partial t} dt = \rho V|_0^T - \int_0^T \frac{\partial \rho}{\partial t} V dt = - \int_0^T \frac{\partial \rho}{\partial t} V dt
$$

since at the boundaries $0$ and $T$, $V = 0$. Hence

$$
I_1 = - \int_0^T \int_0^R \frac{\partial \rho}{\partial t} V(x, t) dt dx. \quad (3)
$$

2.1.2 Evaluation of $I_2$

Change the order of integration in $I_2$ and write it as

$$
I_2 = \int_0^T \int_R \rho u \frac{\partial V}{\partial x} dx dt.
$$

Use integration by parts on the integrand, with $u = \rho u, v' = \frac{\partial V}{\partial x}$ so that $u' = \frac{\partial (\rho u)}{\partial x}, v = V$

$$
\int_R \rho u \frac{\partial V}{\partial x} dx = \rho u V|_R - \int_R \frac{\partial (\rho u)}{\partial x} V dx.
$$

Hence the integral can be evaluated as

$$
I_2 = - \int_0^T \int_R \frac{\partial (\rho u)}{\partial x} V(x, t) dx dt \quad (4)
$$

$$
= - \int_0^T \int_R \frac{\partial (\rho u)}{\partial x} V(x, t) dt dx.
$$

2.1.3 Evaluation of $I_3$

Finally, the evaluation of the integrand of $I_3$ requires the application of integration by parts on $x$ twice. This is because in the integrand we want to get rid of the $\frac{\partial^2 V}{\partial x^2}$ term and end up with $V(x, t)$ only. Again, change the order of integration and write $I_3$ as

$$
\frac{1}{2} \int_0^T \int_R \rho \sigma^2 \frac{\partial^2 V}{\partial x^2} dx dt.
$$
For the first integration by parts use $u = \rho \sigma^2$, $v' = \frac{\partial^2 V}{\partial x^2}$ so that $u' = \frac{\partial (\rho \sigma^2)}{\partial x}$ and $v = \frac{\partial V}{\partial x}$. Hence the integrand can be written

$$
\int_{\mathbb{R}} \rho \sigma^2 \frac{\partial^2 V}{\partial x^2} \, dx = \rho \sigma^2 \frac{\partial V}{\partial x} \bigg|_{\mathbb{R}} - \int_{\mathbb{R}} \frac{\partial (\rho \sigma^2)}{\partial x} \frac{\partial V}{\partial x} \, dx
$$

$$
= - \int_{\mathbb{R}} \frac{\partial (\rho \sigma^2)}{\partial x} \frac{\partial V}{\partial x} \, dx.
$$

Apply integration by parts again, with $u = \frac{\partial (\rho \sigma^2)}{\partial x}$, $v' = \frac{\partial V}{\partial x}$, $u'' = \frac{\partial^2 (\rho \sigma^2)}{\partial x^2}$, $v = V$

$$
- \int_{\mathbb{R}} \frac{\partial (\rho \sigma^2)}{\partial x} \frac{\partial V}{\partial x} \, dx
= - \frac{\partial (\rho \sigma^2)}{\partial x} V \bigg|_{\mathbb{R}} + \int_{\mathbb{R}} \frac{\partial^2 (\rho \sigma^2)}{\partial x^2} V \, dx
$$

$$
= \int_{\mathbb{R}} \frac{\partial^2 (\rho \sigma^2)}{\partial x^2} V(x,t) \, dx.
$$

This implies that $I_3$ can be written as

$$
\frac{1}{2} \int_{0}^{T} \int_{\mathbb{R}} \frac{\partial^2 (\rho \sigma^2)}{\partial x^2} V \, dt \, dx = \frac{1}{2} \int_{\mathbb{R}} \int_{0}^{T} \frac{\partial^2 (\rho \sigma^2)}{\partial x^2} V(x,t) \, dt \, dx.
$$

(5)

2.1.4 Obtaining the Equation

Substitute equations (3), (4), and (5) into equation (2)

$$
E[V(X_T, T)] - V(X_0, 0)
$$

$$
= - \int_{\mathbb{R}} \int_{0}^{T} \frac{\partial \rho}{\partial t} V(x,t) \, dt \, dx - \int_{\mathbb{R}} \int_{0}^{T} \frac{\partial (\rho \mu)}{\partial x} V(x,t) \, dt \, dx
$$

$$
+ \frac{1}{2} \int_{\mathbb{R}} \int_{0}^{T} \frac{\partial^2 (\rho \sigma^2)}{\partial x^2} V(x,t) \, dt \, dx
$$

$$
= \int_{\mathbb{R}} \int_{0}^{T} V(x,t) \left[ - \frac{\partial \rho}{\partial t} - \frac{\partial (\rho \mu)}{\partial x} + \frac{1}{2} \frac{\partial^2 (\rho \sigma^2)}{\partial x^2} \right] \, dt \, dx.
$$

Since $V(X_t, t) = 0$ for $t \notin (0, T)$ we have $V(X_T, T) = V(X_0, 0) = 0$ so that $E[V(X_T, T)] - V(X_0) = 0$. This implies that the portion of the integrand in the brackets is zero

$$
- \frac{\partial \rho}{\partial t} - \frac{\partial (\rho \mu)}{\partial x} + \frac{1}{2} \frac{\partial^2 (\rho \sigma^2)}{\partial x^2} = 0
$$

from which the Fokker-Planck equation can be obtained

$$
\frac{\partial \rho}{\partial t} = - \frac{\partial (\rho \mu)}{\partial x} + \frac{1}{2} \frac{\partial^2 (\rho \sigma^2)}{\partial x^2}.
$$