

Derivation of Local Volatility

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The derivation of local volatility is outlined in many papers and textbooks (such as the one by Jim Gatheral [1]), but in the derivations many steps are left out. In this Note we provide two derivations of local volatility.

1. The derivation by Dupire [2] that uses the Fokker-Planck equation.
2. The derivation by Derman *et al.* [3] of local volatility as a conditional expectation.

We also present the derivation of local volatility from Black-Scholes implied volatility, outlined in [1]. We will derive the following three equations that involve local volatility $\sigma = \sigma(S_t, t)$ or local variance $v_L = \sigma^2$.

1. The Dupire equation in its most general form (appears in [1] on page 9)

$$\frac{\partial C}{\partial T} = \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} + (r_T - q_T) \left(C - K \frac{\partial C}{\partial K} \right). \quad (1)$$

2. The equation by Derman *et al.* [3] for local volatility as a conditional expected value (appears with $q_T = 0$ in [3])

$$\frac{\partial C}{\partial T} = -K(r_T - q_T) \frac{\partial C}{\partial K} - q_T C + \frac{1}{2} K^2 E[\sigma_T^2 | S_T = K] \frac{\partial^2 C}{\partial K^2}. \quad (2)$$

3. Local volatility as a function of Black-Scholes implied volatility, $\Sigma = \Sigma(K, T)$ (appears in [1]) expressed here as the local variance v_L

$$v_L = \frac{\frac{\partial w}{\partial T}}{\left[1 - \frac{y}{w} \frac{\partial w}{\partial y} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2} + \frac{1}{4} \left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w} \right) \left(\frac{\partial w}{\partial y} \right)^2 \right]}. \quad (3)$$

where $w = \Sigma(K, T)^2 T$ is the Black-Scholes total implied variance and $y = \ln \frac{K}{F_T}$ where $F_T = \exp\left(\int_0^T \mu_t dt\right)$ is the forward price with $\mu_t = r_t - q_t$ (risk free rate minus dividend yield). Alternatively, local volatility can also be expressed in terms of Σ as

$$\frac{\Sigma^2 + 2\Sigma T \left[\frac{\partial \Sigma}{\partial T} + (r_T - q_T) K \frac{\partial \Sigma}{\partial K} \right]}{\left(1 + \frac{Ky}{\Sigma} \frac{\partial \Sigma}{\partial K} \right)^2 + K\Sigma T \left[\frac{\partial \Sigma}{\partial K} - \frac{1}{4} K\Sigma T \left(\frac{\partial \Sigma}{\partial K} \right)^2 + K \frac{\partial^2 \Sigma}{\partial K^2} \right]}.$$

Solving for the local variance in Equation (1), we obtain

$$\sigma^2 = \sigma(K, T)^2 = \frac{\frac{\partial C}{\partial T} - (r_T - q_T) \left(C - K \frac{\partial C}{\partial K} \right)}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}. \quad (4)$$

If we set the risk-free rate r_T and the dividend yield q_T each equal to zero, Equations (1) and (2) can each be solved to yield the same equation involving local volatility, namely

$$\sigma^2 = \sigma(K, T)^2 = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2 \frac{\partial^2 C}{\partial K^2}}. \quad (5)$$

The local volatility is then $v_L = \sqrt{\sigma^2(K, T)}$. In this Note the derivation of these equations are all explained in detail.

1 Local Volatility Model for the Underlying

The underlying S_t follows the process

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma(S_t, t) S_t dW_t \\ &= (r_t - q_t) S_t dt + \sigma(S_t, t) S_t dW_t. \end{aligned} \quad (6)$$

We sometimes drop the subscript and write $dS = \mu S dt + \sigma S dW$ where $\sigma = \sigma(S_t, t)$. We need the following preliminaries:

- Discount factor $P(t, T) = \exp\left(-\int_t^T r_s ds\right)$.
- Fokker-Planck equation. Denote by $f(S_t, t)$ the probability density function of the underlying price S_t at time t . Then f satisfies the equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial S} [\mu S f(S, t)] + \frac{1}{2} \frac{\partial^2}{\partial S^2} [\sigma^2 S^2 f(S, t)]. \quad (7)$$

- Time- t price of European call with strike K , denoted $C = C(S_t, K)$

$$\begin{aligned} C &= P(t, T) E[(S_T - K)^+] \\ &= P(t, T) E[(S_T - K) \mathbf{1}_{(S_T > K)}] \\ &= P(t, T) \int_K^\infty (S_T - K) f(S, T) dS. \end{aligned} \quad (8)$$

where $\mathbf{1}_{(S_T > K)}$ is the Heaviside function and where $E[\cdot] = E[\cdot | \mathcal{F}_t]$. In the all the integrals in this Note, since the expectations are taken for the underlying price at $t = T$ it is understood that $S = S_T$, $f(S, T) = f(S_T, T)$ and $dS = dS_T$. We sometimes omit the subscript for notational convenience.

2 Derivation of the General Dupire Equation (1)

2.1 Required Derivatives

We need the following derivatives of the call $C(S_t, t)$.

- First derivative with respect to strike

$$\begin{aligned}\frac{\partial C}{\partial K} &= P(t, T) \int_K^\infty \frac{\partial}{\partial K} (S_T - K) f(S, T) dS \\ &= -P(t, T) \int_K^\infty f(S, T) dS.\end{aligned}\quad (9)$$

- Second derivative with respect to strike

$$\begin{aligned}\frac{\partial^2 C}{\partial K^2} &= -P(t, T) [f(S, T)]_{S=K}^{S=\infty} \\ &= P(t, T) f(K, T).\end{aligned}\quad (10)$$

We have assumed that $\lim_{S \rightarrow \infty} f(S, T) = 0$.

- First derivative with respect to maturity—use the chain rule

$$\begin{aligned}\frac{\partial C}{\partial T} &= \frac{\partial C}{\partial T} P(t, T) \times \int_K^\infty (S_T - K) f(S, T) dS + \\ &P(t, T) \times \int_K^\infty (S_T - K) \frac{\partial}{\partial T} [f(S, T)] dS.\end{aligned}\quad (11)$$

Note that $\frac{\partial P}{\partial T} = -r_T P(t, T)$ so we can write (11)

$$\frac{\partial C}{\partial T} = -r_T C + P(t, T) \int_K^\infty (S_T - K) \frac{\partial}{\partial T} [f(S, T)] dS.\quad (12)$$

2.2 Main Equation

In Equation (12) substitute the Fokker-Planck equation (7) for $\frac{\partial f}{\partial t}$ at $t = T$

$$\begin{aligned}\frac{\partial C}{\partial T} + r_T C &= P(t, T) \int_K^\infty (S_T - K) \times \\ &\left\{ -\frac{\partial}{\partial S} [\mu_T S f(S, T)] + \frac{1}{2} \frac{\partial^2}{\partial S^2} [\sigma^2 S^2 f(S, T)] \right\} dS.\end{aligned}\quad (13)$$

This is the main equation we need because it is from this equation that the Dupire local volatility is derived. In Equation (13) have two integrals to evaluate

$$\begin{aligned}I_1 &= \mu_T \int_K^\infty (S_T - K) \frac{\partial}{\partial S} [S f(S, T)] dS, \\ I_2 &= \int_K^\infty (S_T - K) \frac{\partial^2}{\partial S^2} [\sigma^2 S^2 f(S, T)] dS.\end{aligned}\quad (14)$$

Before evaluating these two integrals we need the following two identities.

2.3 Two Useful Identities

2.3.1 First Identity

From the call price Equation (8), we obtain

$$\begin{aligned} \frac{C}{P(t, T)} &= \int_K^\infty (S_T - K) f(S, T) dS \\ &= \int_K^\infty S_T f(S, T) dS - K \int_K^\infty f(S, T) dS. \end{aligned} \quad (15)$$

From the expression for $\frac{\partial C}{\partial K}$ in Equation (9) we obtain

$$\int_K^\infty f(S, T) dS = -\frac{1}{P(t, T)} \frac{\partial C}{\partial K}.$$

Substitute back into Equation (15) and re-arrange terms to obtain the first identity

$$\int_K^\infty S_T f(S, T) dS = \frac{C}{P(t, T)} - \frac{K}{P(t, T)} \frac{\partial C}{\partial K}. \quad (16)$$

2.3.2 Second Identity

We use the expression for $\frac{\partial^2 C}{\partial K^2}$ in Equation (10) to obtain the second identity

$$f(K, T) = \frac{1}{P(t, T)} \frac{\partial^2 C}{\partial K^2}. \quad (17)$$

2.4 Evaluating the Integrals

We can now evaluate the integrals I_1 and I_2 defined in Equation (14).

2.4.1 First integral

Use integration by parts with $u = S_T - K$, $u' = 1$, $v' = \frac{\partial}{\partial S} [Sf(S, T)]$, $v = Sf(S, T)$

$$\begin{aligned} I_1 &= [\mu_T (S_T - K) Sf(S, T)]_{S=K}^{S=\infty} - \mu_T \int_K^\infty Sf(S, T) dS \\ &= [0 - 0] - \mu_T \int_K^\infty Sf(S, T) dS. \end{aligned}$$

We have assumed $\lim_{S \rightarrow \infty} (S - K) Sf(S, T) = 0$. Substitute the first identity (16) to obtain the first integral I_1

$$I_1 = \frac{-\mu_T C}{P(t, T)} + \frac{\mu_T K}{P(t, T)} \frac{\partial C}{\partial K}. \quad (18)$$

2.4.2 Second integral

Use integration by parts with $u = S_T - K$, $u' = 1$, $v' = \frac{\partial^2}{\partial S^2} [\sigma^2 S^2 f(S, T)]$, $v = \frac{\partial}{\partial S} [\sigma^2 S^2 f(S, T)]$

$$\begin{aligned} I_2 &= \left[(S_T - K) \frac{\partial}{\partial S} \{ \sigma^2 S^2 f(S, T) \} \right]_{S=K}^{S=\infty} - \int_K^\infty \frac{\partial}{\partial S} [\sigma^2 S^2 f(S, T)] dS \\ &= [0 - 0] - [\sigma^2 S^2 f(S, T)]_{S=K}^{S=\infty} \\ &= \sigma^2 K^2 f(K, T) \end{aligned}$$

where $\sigma^2 = \sigma(K, T)^2$. We have assumed that $\lim_{S \rightarrow \infty} \frac{\partial}{\partial S} \{ \sigma^2 S^2 f(S, T) \} = 0$. Substitute the second identity (17) for $f(K, T)$ to obtain the second integral I_2

$$I_2 = \frac{\sigma^2 K^2}{P(t, T)} \frac{\partial^2 C}{\partial K^2}. \quad (19)$$

2.5 Obtaining the Dupire Equation

We can now evaluate the main Equation (13) which we write as

$$\frac{\partial C}{\partial T} + r_T C = P(t, T) \left[-I_1 + \frac{1}{2} I_2 \right].$$

Substitute for I_1 from Equation (18) and for I_2 from Equation (19)

$$\frac{\partial C}{\partial T} + r_T C = \mu_T C - \mu_T K \frac{\partial C}{\partial K} + \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 C}{\partial K^2}$$

Substitute for $\mu_T = r_T - q_T$ (risk free rate minus dividend yield) to obtain the Dupire equation (1)

$$\frac{\partial C}{\partial T} = \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} + (r_T - q_T) \left(C - K \frac{\partial C}{\partial K} \right).$$

Solve for $\sigma^2 = \sigma(K, T)^2$ to obtain the Dupire local variance in its general form

$$\sigma(K, T)^2 = \frac{\frac{\partial C}{\partial T} + q_T C + (r_T - q_T) K \frac{\partial C}{\partial K}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}$$

Dupire [2] assumes zero interest rates and zero dividend yield. Hence $r_T = q_T = 0$ so that the underlying process is $dS_t = \sigma(S_t, t) S_t dW_t$. We obtain

$$\sigma(K, T)^2 = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}.$$

which is Equation (5).

3 Derivation of Local Volatility as an Expected Value, Equation (2)

We need the following preliminaries, all of which are easy to show

| | |
|---|---|
| $\frac{\partial}{\partial S}(S - K)^+ = \mathbf{1}_{(S > K)}$ | $\frac{\partial}{\partial S} \mathbf{1}_{(S > K)} = \delta(S - K)$ |
| $\frac{\partial}{\partial K}(S - K)^+ = -\mathbf{1}_{(S > K)}$ | $\frac{\partial}{\partial K} \mathbf{1}_{(S > K)} = -\delta(S - K)$ |
| $\frac{\partial C}{\partial K} = -P(t, T)E[\mathbf{1}_{(S > K)}]$ | $\frac{\partial^2 C}{\partial K^2} = P(t, T)E[\delta(S - K)]$ |

In the table, $\delta(\cdot)$ denotes the Dirac delta function. Now define the function $f(S_T, T)$ as

$$f(S_T, T) = P(t, T)(S_T - K)^+.$$

Recall the process for S_t is given by Equation (6). By Itô's Lemma, f follows the process

$$df = \left[\frac{\partial f}{\partial T} + \mu_T S_T \frac{\partial f}{\partial S_T} + \frac{1}{2} \sigma_T^2 S_T^2 \frac{\partial^2 f}{\partial S_T^2} \right] dT + \left[\sigma_T S_T \frac{\partial f}{\partial S_T} \right] dW_T. \quad (20)$$

Now the partial derivatives are

$$\begin{aligned} \frac{\partial f}{\partial T} &= -r_T P(t, T)(S_T - K)^+, \\ \frac{\partial f}{\partial S_T} &= P(t, T) \mathbf{1}_{(S_T > K)}, \\ \frac{\partial^2 f}{\partial S_T^2} &= P(t, T) \delta(S_T - K). \end{aligned}$$

Substitute them into Equation (20)

$$\begin{aligned} df &= P(t, T) \times \quad (21) \\ &\quad \left[-r_T (S_T - K)^+ + \mu_T S_T \mathbf{1}_{(S_T > K)} + \frac{1}{2} \sigma_T^2 S_T^2 \delta(S_T - K) \right] dT \\ &\quad + P(t, T) \left[\sigma_T S_T \mathbf{1}_{(S_T > K)} \right] dW_T \end{aligned}$$

Consider the first two terms of (21), which can be written as

$$\begin{aligned} -r_T (S_T - K)^+ + \mu_T S_T \mathbf{1}_{(S_T > K)} &= -r_T (S_T - K) \mathbf{1}_{(S_T > K)} + \mu_T S_T \mathbf{1}_{(S_T > K)} \\ &= r_T K \mathbf{1}_{(S_T > K)} - q_T S_T \mathbf{1}_{(S_T > K)}. \end{aligned}$$

When we take the expected value of Equation (21), the stochastic term drops out since $E[dW_T] = 0$. Hence we can write the expected value of (21) as

$$\begin{aligned} dC &= E[df] \quad (22) \\ &= P(t, T)E \left[r_T K \mathbf{1}_{(S_T > K)} - q_T S_T \mathbf{1}_{(S_T > K)} + \frac{1}{2} \sigma_T^2 S_T^2 \delta(S_T - K) \right] dT \end{aligned}$$

so that

$$\frac{dC}{dT} = P(t, T)E \left[r_T K \mathbf{1}_{(S_T > K)} - q_T S_T \mathbf{1}_{(S_T > K)} + \frac{1}{2} \sigma_T^2 S_T^2 \delta(S_T - K) \right]. \quad (23)$$

Using the second line in Equation (8), we can write

$$P(t, T)E [S_T \mathbf{1}_{(S_T > K)}] = C + KP(t, T)E [\mathbf{1}_{(S_T > K)}]$$

so Equation (23) becomes

$$\begin{aligned} \frac{dC}{dT} &= KP(t, T)r_T E[\mathbf{1}_{(S_T > K)}] - q_T (C + KP(t, T)E [\mathbf{1}_{(S_T > K)}]) \\ &\quad + \frac{1}{2}P(t, T)E [\sigma_T^2 S_T^2 \delta(S_T - K)] \\ &= -K(r_T - q_T) \frac{\partial C}{\partial K} - q_T C + \frac{1}{2}P(t, T)E [\sigma_T^2 S_T^2 \delta(S_T - K)] \end{aligned} \quad (24)$$

where we have substituted $-\frac{\partial C}{\partial K}$ for $P(t, T)E[\mathbf{1}_{(S_T > K)}]$. The last term in the last line of Equation (24) can be written

$$\begin{aligned} \frac{1}{2}P(t, T)E [\sigma_T^2 S_T^2 \delta(S_T - K)] &= \frac{1}{2}P(t, T)E [\sigma_T^2 S_T^2 | S_T = K] E[\delta(S_T - K)] \\ &= \frac{1}{2}P(t, T)E [\sigma_T^2 | S_T = K] K^2 E[\delta(S_T - K)] \\ &= \frac{1}{2}E [\sigma_T^2 | S_T = K] K^2 \frac{\partial^2 C}{\partial K^2} \end{aligned}$$

where we have substituted $\frac{\partial^2 C}{\partial K^2}$ for $P(t, T)E[\delta(S_T - K)]$. We obtain the final result, Equation (2)

$$\frac{\partial C}{\partial T} = -K(r_T - q_T) \frac{\partial C}{\partial K} - q_T C + \frac{1}{2}K^2 E [\sigma_T^2 | S_T = K] \frac{\partial^2 C}{\partial K^2}.$$

When $r_T = q_T = 0$ we can re-arrange the result to obtain

$$E [\sigma_T^2 | S_T = K] = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2 \frac{\partial^2 C}{\partial K^2}}$$

which, again, is Equation (5). Hence when the dividend and interest rate are both zero, the derivation of local volatility using Dupire's approach and the derivation using conditional expectation produce the same result.

4 Derivation of Local Volatility From Implied Volatility, Equation (3)

To express local volatility in terms of implied volatility, we need the three derivatives $\frac{\partial C}{\partial T}$, $\frac{\partial C}{\partial K}$, and $\frac{\partial^2 C}{\partial K^2}$ that appear in Equation (1), but expressed in terms of

implied volatility. Following Gatheral [1] we define the log-moneyness

$$y = \ln \frac{K}{F_T}$$

where $F_T = S_0 \exp\left(\int_0^T \mu_t dt\right)$ is the forward price ($\mu_t = r_t - q_t$, risk free rate minus dividend yield) and K is the strike price, and the "total" Black-Scholes implied variance

$$w = \Sigma(K, T)^2 T$$

where $\Sigma(K, T)$ is the implied volatility. The Black-Scholes call price can then be written as

$$\begin{aligned} C_{BS}(S_0, K, \Sigma(K, T), T) &= C_{BS}(S_0, F_T e^y, w, T) \\ &= F_T \{N(d_1) - e^y N(d_2)\} \end{aligned} \quad (25)$$

where

$$d_1 = \frac{\ln \frac{S_0}{K} + \int_0^T (r_t - q_t) dt + \frac{w}{2}}{\sqrt{w}} = -yw^{-\frac{1}{2}} + \frac{1}{2}w^{\frac{1}{2}} \quad (26)$$

and $d_2 = d_1 - \sqrt{w} = -yw^{-\frac{1}{2}} - \frac{1}{2}w^{\frac{1}{2}}$.

4.1 The Reparameterized Local Volatility Function

To express the local volatility Equation (1) in terms of y , we note that the market call price is

$$C(S_0, K, T) = C(S_0, F_T e^y, T)$$

and we take derivatives. The first derivative we need is, by the chain rule

$$\frac{\partial C}{\partial y} = \frac{\partial C}{\partial K} \frac{\partial K}{\partial y} = \frac{\partial C}{\partial K} K. \quad (27)$$

The second derivative we need is

$$\begin{aligned} \frac{\partial^2 C}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial C}{\partial K} \right) K + \frac{\partial C}{\partial K} \frac{\partial K}{\partial y} \\ &= \frac{\partial^2 C}{\partial K^2} K^2 + \frac{\partial C}{\partial y}, \end{aligned} \quad (28)$$

since by the chain rule $\frac{\partial A}{\partial y} = \frac{\partial A}{\partial K} \frac{\partial K}{\partial y}$, so that $\frac{\partial}{\partial y} \left(\frac{\partial C}{\partial K} \right) = \frac{\partial^2 C}{\partial K^2} \frac{\partial K}{\partial y} = \frac{\partial^2 C}{\partial K^2} K$. The third derivative we need is

$$\begin{aligned} \frac{\partial C}{\partial T} &= \frac{\partial C}{\partial T} + \frac{\partial C}{\partial K} \frac{\partial K}{\partial T} \\ &= \frac{\partial C}{\partial T} + \frac{\partial C}{\partial K} K \mu_T \\ &= \frac{\partial C}{\partial T} + \frac{\partial C}{\partial y} \mu_T \end{aligned} \quad (29)$$

since $K = S_0 \exp\left(\int_0^T \mu_t dt + y\right)$ so that $\frac{\partial K}{\partial T} = K\mu_T$. Equation (28) implies that

$$\frac{\partial^2 C}{\partial K^2} K^2 = \frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y}.$$

Now we substitute into Equation (1), reproduced here for convenience

$$\begin{aligned} \frac{\partial C}{\partial T} &= \frac{1}{2} \sigma^2 K^2 \frac{\partial^2 C}{\partial K^2} + \mu_T \left(C - K \frac{\partial C}{\partial K} \right) \\ \frac{\partial C}{\partial T} - \frac{\partial C}{\partial y} \mu_T &= \frac{1}{2} \sigma^2 \left(\frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y} \right) + \mu_T \left(C - \frac{\partial C}{\partial y} \right) \end{aligned}$$

which simplifies to

$$\frac{\partial C}{\partial T} = \frac{v_L}{2} \left[\frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y} \right] + \mu_T C \quad (30)$$

where $v_L = \sigma^2(K, T)$ is the local variance. This is Equation (1.8) of Gatheral [1].

4.2 Three Useful Identities

Before expression the local volatility Equation (1) in terms of implied volatility, we first derive three identities used by Gatheral [1] that help in this regard. We use the fact that the derivatives of the standard normal cdf and pdf are, using the chain rule, $N'(x) = n(x)x'$ and $n'(x) = -xn(x)x'$. We also use the relation

$$\begin{aligned} n(d_1) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_2 + \sqrt{w})^2} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_2^2 + 2d_2\sqrt{w} + w)} \\ &= n(d_2) e^{-d_2\sqrt{w} - \frac{1}{2}w} \\ &= n(d_2) e^y. \end{aligned}$$

From Equation (25) the first derivative with respect to w is

$$\begin{aligned} \frac{\partial C_{BS}}{\partial w} &= F_T [n(d_1)d_{1w} - e^y n(d_2)d_{2w}] \\ &= F_T \left[n(d_2)e^y \left(d_{2w} + \frac{1}{2}w^{-\frac{1}{2}} \right) - e^y n(d_2)d_{2w} \right] \\ &= \frac{1}{2} F_T e^y \left[n(d_2)w^{-\frac{1}{2}} \right] \end{aligned}$$

where d_{1w} is the first derivative of d_1 with respect to w and similarly for d_2 . The second derivative is

$$\begin{aligned}
\frac{\partial^2 C_{BS}}{\partial w^2} &= \frac{1}{2} F_T e^y \left[-n(d_2) d_2 d_{2w} w^{-\frac{1}{2}} - \frac{1}{2} n(d_2) w^{-\frac{3}{2}} \right] \\
&= \frac{1}{2} F_T e^y n(d_2) w^{-\frac{1}{2}} \left[-d_2 d_{2w} - \frac{1}{2} w^{-1} \right] \\
&= \frac{\partial C_{BS}}{\partial w} \left[\left(y w^{-\frac{1}{2}} + \frac{1}{2} w^{\frac{1}{2}} \right) \left(\frac{1}{2} y w^{-\frac{3}{2}} - \frac{1}{4} w^{-\frac{1}{2}} \right) - \frac{1}{2} w^{-1} \right] \\
&= \frac{\partial C_{BS}}{\partial w} \left[-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2} \right].
\end{aligned} \tag{31}$$

This is the first identity we need. The second identity we need is

$$\begin{aligned}
\frac{\partial^2 C_{BS}}{\partial w \partial y} &= \frac{1}{2} F_T w^{-\frac{1}{2}} \frac{\partial}{\partial y} [e^y n(d_2)] \\
&= \frac{1}{2} F_T w^{-\frac{1}{2}} [e^y n(d_2) - e^y n(d_2) d_2 d_{2y}] \\
&= \frac{\partial C_{BS}}{\partial w} [1 - d_2 d_{2y}] \\
&= \frac{\partial C_{BS}}{\partial w} \left(\frac{1}{2} - \frac{y}{w} \right)
\end{aligned} \tag{32}$$

where $d_{2y} = -w^{-\frac{1}{2}}$ is the first derivative of d_2 with respect to y . To obtain the third identity, consider the derivative

$$\begin{aligned}
\frac{\partial C_{BS}}{\partial y} &= F_T [n(d_1) d_{1y} - e^y N(d_2) - e^y n(d_2) d_{2y}] \\
&= F_T e^y [n(d_2) d_{1y} - N(d_2) - n(d_2) d_{2y}] \\
&= -F_T e^y N(d_2).
\end{aligned}$$

The third identity we need is

$$\begin{aligned}
\frac{\partial^2 C_{BS}}{\partial y^2} &= -F_T [e^y N(d_2) + e^y n(d_2) d_{2y}] \\
&= -F_T e^y N(d_2) + F_T e^y n(d_2) w^{-\frac{1}{2}} \\
&= \frac{\partial C_{BS}}{\partial y} + 2 \frac{\partial C_{BS}}{\partial w}.
\end{aligned} \tag{33}$$

We are now ready for the main derivation of this section.

4.3 Local Volatility in Terms of Implied Volatility

We note that when the market price $C(S_0, K, T)$ is equal to the Black-Scholes price with the implied volatility $\Sigma(K, T)$ as the input to volatility

$$C(S_0, K, T) = C_{BS}(S_0, K, \Sigma(K, T), T). \tag{34}$$

We can also reparameterize the Black-Scholes price in terms of the total implied volatility $w = \Sigma(K, T)^2 T$ and $K = F_T e^y$. Since w depends on K and K depends on y , we have that $w = w(y)$ and we can write

$$C(S_0, K, T) = C_{BS}(S_0, F_T e^y, w(y), T). \quad (35)$$

We need derivatives of the market call price $C(S_0, K, T)$ in terms of the Black-Scholes call price $C_{BS}(S_0, F_T e^y, w(y), T)$. From Equation (35), the first derivative we need is

$$\begin{aligned} \frac{\partial C}{\partial y} &= \frac{\partial C_{BS}}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial y} \\ &= a(w, y) + b(w, y)c(y). \end{aligned} \quad (36)$$

It is easier to visualize the second derivative we need, $\frac{\partial^2 C}{\partial y^2}$, when we express the partials in $\frac{\partial C}{\partial y}$ as a, b , and c .

$$\begin{aligned} \frac{\partial^2 C}{\partial y^2} &= \frac{\partial a}{\partial y} + \frac{\partial a}{\partial w} \frac{\partial w}{\partial y} + b(w, y) \frac{\partial c}{\partial y} + \left[\frac{\partial b}{\partial y} + \frac{\partial b}{\partial w} \frac{\partial w}{\partial y} \right] c(y) \\ &= \frac{\partial^2 C_{BS}}{\partial y^2} + \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial y^2} + \left[\frac{\partial^2 C_{BS}}{\partial w \partial y} + \frac{\partial^2 C_{BS}}{\partial w^2} \frac{\partial w}{\partial y} \right] \frac{\partial w}{\partial y} \\ &= \frac{\partial^2 C_{BS}}{\partial y^2} + 2 \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 C_{BS}}{\partial w^2} \left(\frac{\partial w}{\partial y} \right)^2. \end{aligned} \quad (37)$$

The third derivative we need is

$$\begin{aligned} \frac{\partial C}{\partial T} &= \frac{\partial C_{BS}}{\partial T} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T} \\ &= \mu_T C + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T}. \end{aligned} \quad (38)$$

Gatheral explains that the second equality follows because the only explicit dependence of C_{BS} on T is through the forward price F_T , even though C_{BS} depends implicitly on T through y and w . The reparameterized Dupire equation (30) is reproduced here for convenience

$$\frac{\partial C}{\partial T} = \frac{v_L}{2} \left[\frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y} \right] + \mu_T C.$$

We substitute for $\frac{\partial C}{\partial T}$, $\frac{\partial^2 C}{\partial y^2}$, and $\frac{\partial C}{\partial y}$ from Equations (38), (37), and (36) respectively and cancel $\mu_T C$ from both sides to obtain

$$\begin{aligned} \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T} &= \frac{v_L}{2} \left[\frac{\partial^2 C_{BS}}{\partial y^2} + 2 \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 C_{BS}}{\partial w^2} \left(\frac{\partial w}{\partial y} \right)^2 \right. \\ &\quad \left. - \frac{\partial C_{BS}}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial y} \right]. \end{aligned} \quad (39)$$

Now substitute for $\frac{\partial^2 C_{BS}}{\partial w^2}$, $\frac{\partial^2 C_{BS}}{\partial w \partial y}$, and $\frac{\partial^2 C_{BS}}{\partial y^2}$ from the identities in Equations (31), (32), and (33) respectively, the idea being to end up with terms involving $\frac{\partial C_{BS}}{\partial w}$ on the right hand side of Equation (39) that can be factored out.

$$\begin{aligned} \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T} &= \frac{v_L}{2} \frac{\partial C_{BS}}{\partial w} \left[2 + 2 \left(\frac{1}{2} - \frac{y}{w} \right) \frac{\partial w}{\partial y} + \left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w} \right) \left(\frac{\partial w}{\partial y} \right)^2 \right. \\ &\quad \left. + \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial y} \right]. \end{aligned}$$

Remove the factor $\frac{\partial C_{BS}}{\partial w}$ from both sides and simplify to obtain

$$\frac{\partial w}{\partial T} = v_L \left[1 - \frac{y}{w} \frac{\partial w}{\partial y} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2} + \frac{1}{4} \left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w} \right) \left(\frac{\partial w}{\partial y} \right)^2 \right].$$

Solve for v_L to obtain the final expression for the local volatility expressed in terms of implied volatility $w = \Sigma(K, T)^2 T$ and the log-moneyness $y = \ln \frac{K}{F_T}$

$$v_L = \frac{\frac{\partial w}{\partial T}}{\left[1 - \frac{y}{w} \frac{\partial w}{\partial y} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2} + \frac{1}{4} \left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w} \right) \left(\frac{\partial w}{\partial y} \right)^2 \right]}.$$

4.4 Alternate Derivation

In this derivation we express the derivatives $\frac{\partial C}{\partial K}$, $\frac{\partial^2 C}{\partial K^2}$, and $\frac{\partial C}{\partial T}$ in the Dupire equation (1) in terms of y and $w = w(y)$, but we substitute these derivatives directly in Equation (1) rather than in (30). This means that we take derivatives with respect to K and T , rather than with y and T . Recall that from Equation (35), the market call price is equal to the Black-Scholes call price with implied volatility as input

$$C(S_0, K, T) = C_{BS}(S_0, F_T e^y, w(y), T).$$

Recall also that from Equation (25) the Black-Scholes call price reparameterized in terms of y and w is

$$C_{BS}(S_0, F_T e^y, w(y), T) = F_T \{N(d_1) - e^y N(d_2)\}$$

where d_1 is given in Equation (26), and where $d_2 = d_1 - \sqrt{w}$. The first derivative

we need is

$$\begin{aligned} \frac{\partial C}{\partial K} &= \frac{\partial C_{BS}}{\partial y} \frac{\partial y}{\partial K} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial K} \\ &= \frac{1}{K} \frac{\partial C_{BS}}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial K}. \end{aligned} \tag{40}$$

The second derivative is

$$\begin{aligned} \frac{\partial^2 C}{\partial K^2} &= -\frac{1}{K^2} \frac{\partial C_{BS}}{\partial y} + \frac{1}{K} \frac{\partial}{\partial K} \left(\frac{\partial C_{BS}}{\partial y} \right) \\ &\quad + \frac{\partial}{\partial K} \left(\frac{\partial C_{BS}}{\partial w} \right) \frac{\partial w}{\partial K} + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial K^2} \end{aligned} \quad (41)$$

Let $A = \frac{\partial C}{\partial y}$ for notational convenience. Then $\frac{\partial}{\partial K} \left(\frac{\partial C}{\partial y} \right) = \frac{\partial A}{\partial K}$ and

$$\begin{aligned} \frac{\partial}{\partial K} \left(\frac{\partial C_{BS}}{\partial y} \right) &= \frac{\partial A}{\partial K} \\ &= \frac{\partial A}{\partial y} \frac{\partial y}{\partial K} + \frac{\partial A}{\partial w} \frac{\partial w}{\partial K} \\ &= \frac{\partial^2 C_{BS}}{\partial y^2} \frac{1}{K} + \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial K}. \end{aligned} \quad (42)$$

Similarly

$$\frac{\partial}{\partial K} \left(\frac{\partial C_{BS}}{\partial w} \right) = \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{1}{K} + \frac{\partial^2 C_{BS}}{\partial w^2} \frac{\partial w}{\partial K}. \quad (43)$$

Substituting Equations (42) and (43) into Equation (41) produces

$$\begin{aligned} \frac{\partial^2 C}{\partial K^2} &= -\frac{1}{K^2} \frac{\partial C_{BS}}{\partial y} + \frac{1}{K} \left(\frac{\partial^2 C_{BS}}{\partial y^2} \frac{1}{K} + \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial K} \right) \\ &\quad + \left(\frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{1}{K} + \frac{\partial^2 C_{BS}}{\partial w^2} \frac{\partial w}{\partial K} \right) \frac{\partial w}{\partial K} + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial K^2} \\ &= \frac{1}{K^2} \left(\frac{\partial^2 C_{BS}}{\partial y^2} - \frac{\partial C_{BS}}{\partial y} \right) + \frac{2}{K} \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial K} \\ &\quad + \frac{\partial^2 C_{BS}}{\partial w^2} \left(\frac{\partial w}{\partial K} \right)^2 + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial K^2}. \end{aligned} \quad (44)$$

The third derivative we need is

$$\begin{aligned} \frac{\partial C}{\partial T} &= \frac{\partial C_{BS}}{\partial T} + \frac{\partial C_{BS}}{\partial y} \frac{\partial y}{\partial T} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T} \\ &= \mu_T C_{BS} + \frac{\partial C_{BS}}{\partial y} \mu_T + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T}, \end{aligned} \quad (45)$$

again using the fact that $\frac{\partial C_{BS}}{\partial T}$ depends explicitly on T only through F_T . Now substitute for $\frac{\partial C}{\partial K}$, $\frac{\partial^2 C}{\partial K^2}$, and $\frac{\partial C}{\partial T}$ from Equations (40), (44), and (45) respectively into Equation (4) for Dupire local variance, reproduced here for convenience.

$$\sigma^2 = \frac{\frac{\partial C}{\partial T} - \mu_T [C_{BS} - K \frac{\partial C}{\partial K}]}{\frac{1}{2} K^2 \frac{\partial^2 C}{\partial K^2}}.$$

We obtain, after applying the three useful identities in Section 4.2,

$$\sigma^2 = \frac{\mu_T C_{BS} + \frac{\partial C_{BS}}{\partial y} \mu_T + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial T} - \mu_T \left[C_{BS} - K \left(\frac{1}{K} \frac{\partial C_{BS}}{\partial y} + \frac{\partial C_{BS}}{\partial w} \frac{\partial w}{\partial K} \right) \right]}{\frac{1}{2} K^2 \left[\frac{1}{K^2} \left(\frac{\partial^2 C_{BS}}{\partial y^2} - \frac{\partial C_{BS}}{\partial y} \right) + \frac{2}{K} \frac{\partial^2 C_{BS}}{\partial y \partial w} \frac{\partial w}{\partial K} + \frac{\partial^2 C_{BS}}{\partial w^2} \left(\frac{\partial w}{\partial K} \right)^2 + \frac{\partial C_{BS}}{\partial w} \frac{\partial^2 w}{\partial K^2} \right]}.$$

Applying the three useful identities in Section 4.2 allows the term $\frac{\partial C_{BS}}{\partial w}$ to be factored out of the numerator and denominator. The last equation becomes

$$\sigma^2 = \frac{\left[\frac{\partial w}{\partial T} + \mu_T K \frac{\partial w}{\partial K} \right]}{\frac{1}{2} K^2 \left[\frac{2}{K^2} + \frac{2}{K} \left(\frac{1}{2} - \frac{y}{w} \right) \frac{\partial w}{\partial K} + \left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2} \right) \left(\frac{\partial w}{\partial K} \right)^2 + \frac{\partial^2 w}{\partial K^2} \right]}. \quad (46)$$

Equation (46) can be simplified by considering deriving the partial derivatives of the Black-Scholes total implied variance, $w = \Sigma(K, T)^2 T$. We have $\frac{\partial w}{\partial T} = 2\Sigma T \frac{\partial \Sigma}{\partial T} + \Sigma^2$, $\frac{\partial w}{\partial K} = 2\Sigma T \frac{\partial \Sigma}{\partial K}$, and $\frac{\partial^2 w}{\partial K^2} = 2T \left[\left(\frac{\partial \Sigma}{\partial K} \right)^2 + \Sigma \frac{\partial^2 \Sigma}{\partial K^2} \right]$. Substitute into Equation (46). The numerator in Equation (46) becomes

$$\Sigma^2 + 2\Sigma T \left(\frac{\partial \Sigma}{\partial T} + \mu_T K \frac{\partial \Sigma}{\partial K} \right) \quad (47)$$

and the denominator becomes

$$1 + 2K\Sigma T \left(\frac{1}{2} - \frac{y}{w} \right) \frac{\partial \Sigma}{\partial K} + 2K^2 \Sigma^2 T^2 \left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2} \right) \left(\frac{\partial \Sigma}{\partial K} \right)^2 + K^2 T \left[\left(\frac{\partial \Sigma}{\partial K} \right)^2 + \Sigma \frac{\partial^2 \Sigma}{\partial K^2} \right].$$

Replacing w with $\Sigma^2 T$ everywhere in the denominator produces

$$\begin{aligned} & 1 + 2K\Sigma T \left(\frac{1}{2} - \frac{y}{\Sigma^2 T} \right) \frac{\partial \Sigma}{\partial K} + 2K^2 \Sigma^2 T^2 \left(-\frac{1}{8} - \frac{1}{2\Sigma^2 T} + \frac{y^2}{2\Sigma^4 T^2} \right) \left(\frac{\partial \Sigma}{\partial K} \right)^2 \\ & + K^2 T \left[\left(\frac{\partial \Sigma}{\partial K} \right)^2 + \Sigma \frac{\partial^2 \Sigma}{\partial K^2} \right] \\ = & 1 + K\Sigma T \frac{\partial \Sigma}{\partial K} - \frac{2Ky}{\Sigma} \frac{\partial \Sigma}{\partial K} - \frac{K^2 \Sigma^2 T^2}{4} \left(\frac{\partial \Sigma}{\partial K} \right)^2 + \frac{K^2 y^2}{\Sigma^2} \left(\frac{\partial \Sigma}{\partial K} \right)^2 \\ & + K^2 \Sigma T \frac{\partial^2 \Sigma}{\partial K^2} \\ = & \left(1 - \frac{Ky}{\Sigma} \frac{\partial \Sigma}{\partial K} \right)^2 + \left[1 - 2 \frac{Ky}{\Sigma} \frac{\partial \Sigma}{\partial K} + \left(\frac{Ky}{\Sigma} \frac{\partial \Sigma}{\partial K} \right)^2 \right]. \end{aligned} \quad (48)$$

Substituting the numerator in (47) and the denominator in (48) back to Equation (46), we obtain

$$\frac{\Sigma^2 + 2\Sigma T \left(\frac{\partial \Sigma}{\partial T} + \mu_T K \frac{\partial \Sigma}{\partial K} \right)}{\left(1 + \frac{Ky}{\Sigma} \frac{\partial \Sigma}{\partial K} \right)^2 + K\Sigma T \left[\frac{\partial \Sigma}{\partial K} - \frac{1}{4} K\Sigma T \left(\frac{\partial \Sigma}{\partial K} \right)^2 + K \frac{\partial^2 \Sigma}{\partial K^2} \right]}$$

See also the dissertation by van der Kamp [4] for additional details of this alternate derivation.

References

- [1] Gatheral, J. (2006). *The Volatility Surface: A Practitioner's Guide*. New York, NY: John Wiley & Sons.
- [2] Dupire, B. (1994). "Pricing With a Smile." *Risk* 7, pp. 18-20.
- [3] Derman, E., Kani, I., and M. Kamal (1996). "Trading and Hedging Local Volatility." Goldman Sachs Quantitative Strategies Research Notes.
- [4] van der Kamp, Roel (2009). "Local Volatility Modelling." M.Sc. dissertation, University of Twente, The Netherlands.