# Euler and Milstein Discretization

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"Monte Carlo simulation" in the context of option pricing refers to a set of techniques to generate underlying values–typically stock prices or interest rates–over time. Typically the dynamics of these stock prices and interest rates are assumed to be driven by a continuous-time stochastic process. Simulation, however, is done at discrete time steps. Hence, the first step in any simulation scheme is to find a way to "discretize" a continuous-time process into a discrete time process. In this Note we present two discretization schemes, Euler and Milstein discretization, and illustrate both with the Black-Scholes and the Heston models.

We assume that the stock price $S_t$ is driven by the stochastic differential equation (SDE)

$$dS_t = \mu(S_t, t)\,dt + \sigma(S_t, t)\,dW_t \tag{1}$$

where $W_t$ is Brownian motion. We simulate $S_t$ over the time interval $[0, T]$, which we assume to be is discretized as $0 = t_1 < t_2 < \cdots < t_m = T$, where the time increments are equally spaced with width $dt$. Equally-spaced time increments is primarily used for notational convenience, because it allows us to write $t_i - t_{i-1}$ as simply $dt$. All the results derived with equally-spaced increments are easily generalized to unequal spacing.

Integrating $dS_t$ from $t$ to $t + dt$ produces

$$S_{t+dt} = S_t + \int_t^{t+dt} \mu(S_u, u)\,du + \int_t^{t+dt} \sigma(S_u, u)\,dW_u. \tag{2}$$

Equation (2) is the starting point for any discretization scheme. At time $t$, the value of $S_t$ is known, and we wish to obtain the next value $S_{t+dt}$.

## 1 Euler Scheme

The simplest way to discretize the process in Equation (2) is to use Euler discretization. This is equivalent to approximating the integrals using the left-point rule. Hence the first integral is approximated as the product of the integrand at time $t$, and the integration range $dt$

$$\begin{aligned}
\int_t^{t+dt} \mu(S_u, u)\,du &\approx \mu(S_t, t)\int_t^{t+dt} du \\
&= \mu(S_t, t)\,dt.
\end{aligned}$$

We use the left-point rule since at time $t$ the value $\mu(S_t, t)$ is known. The right-hand rule would require that $\mu(S_{t+dt}, t + dt)$ be known at time $t$. In an identical fashion, the second integral is approximated as

$$\begin{aligned}\int_t^{t+dt} \sigma(S_u, u)\, dW_u &\approx \sigma(S_t, t) \int_t^{t+dt} dW_u \\ &= \sigma(S_t, t)(W_{t+dt} - W_t) \\ &= \sigma(S_t, t)\sqrt{dt}Z,\end{aligned}$$

since $W_{t+dt} - W_t$ and $\sqrt{dt}Z$ are identical in distribution, where $Z$ is a standard normal variable. Hence, Euler discretization of (2) is

$$S_{t+dt} = S_t + \mu(S_t, t)\, dt + \sigma(S_t, t)\sqrt{dt}Z. \tag{3}$$

## 1.1 Euler Scheme for the Black-Scholes Model

The Black-Scholes stock price dynamics under the risk neutral measure are

$$dS_t = rS_t dt + \sigma S_t dW_t. \tag{4}$$

An application of Equation (3) produces Euler discretization for the Black-Scholes model

$$S_{t+dt} = S_t + rS_t dt + \sigma S_t \sqrt{dt}Z. \tag{5}$$

Alternatively, we can generate log-stock prices, and exponentiate the result. By Itō's lemma $\ln S_t$ follows the process

$$d \ln S_t = \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t. \tag{6}$$

Euler discretization via Equation (3) produces

$$\ln S_{t+dt} = \ln S_t + \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma\sqrt{dt}Z$$

so that

$$S_{t+dt} = S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right) dt + \sigma\sqrt{dt}Z\right). \tag{7}$$

where $dt = t_i - t_{i-1}$.

## 1.2 Euler Scheme for the Heston Model

The Heston model is described by the bivariate stochastic process for the stock price $S_t$ and its variance $v_t$

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t dW_{1,t} \\ dv_t &= \kappa(\theta - v_t)\, dt + \sigma\sqrt{v_t} dW_{2,t} \end{aligned} \tag{8}$$

where $E[dW_{1,t} dW_{2,t}] = \rho dt$.

### 1.2.1 Discretization of $v_t$

The SDE for $v_t$ in (8) in integral form is

$$v_{t+dt} = v_t + \int_t^{t+dt} \kappa\left(\theta - v_u\right) du + \int_t^{t+dt} \sigma\sqrt{v_u}dW_{2,u}. \tag{9}$$

The Euler discretization approximates the integrals using the left-point rule

$$\begin{aligned}
\int_t^{t+dt} \kappa\left(\theta - v_u\right) du &\approx \kappa\left(\theta - v_t\right) dt \\
\int_t^{t+dt} \sigma\sqrt{v_u}dW_{2,u} &\approx \sigma\sqrt{v_t}\left(W_{t+dt} - W_t\right) \\
&= \sigma\sqrt{v_t dt}Z_v
\end{aligned}$$

where $Z_v$ is a standard normal random variable. The right hand side involves $(\theta - v_t)$ rather than $(\theta - v_{t+dt})$ since at time $t$ we don't know the value of $v_{t+dt}$. This leaves us with

$$v_{t+dt} = v_t + \kappa\left(\theta - v_t\right) dt + \sigma\sqrt{v_t dt}Z_v.$$

To avoid negative variances, we can replace $v_t$ with $v_t^+ = \max\left(0, v_t\right)$. This is the *full truncation* scheme. The *reflection* scheme replaces $v_t$ with its absolute value $|v_t|$.

### 1.2.2 Process for $S_t$

In a similar fashion, the SDE for $S_t$ in (8) is written in integral form as

$$S_{t+dt} = S_t + r\int_t^{t+dt} S_u du + \int_t^{t+dt} \sqrt{v_u}S_u dW_u.$$

Euler discretization approximates the integrals with the left-point rule

$$\begin{aligned}
\int_t^{t+dt} S_u du &\approx S_t dt \\
\int_t^{t+dt} \sqrt{v_u}S_u dW_{1,u} &\approx \sqrt{v_t}S_t\left(W_{t+dt} - W_t\right) \\
&= \sqrt{v_t dt}S_t Z_s
\end{aligned}$$

where $Z_s$ is a standard normal random variable that has correlation $\rho$ with $Z_v$. We end up with

$$S_{t+dt} = S_t + rS_t dt + \sqrt{v_t dt}S_t Z_s.$$

## 1.3 Process for $\ln S_t$

By Itō's lemma $\ln S_t$ follows the diffusion

$$d\ln S_t = \left(r - \frac{1}{2}v_t\right)dt + \sqrt{v_t}dW_{1,t}$$

or in integral form

$$\ln S_{t+dt} = \ln S_t + \int_0^t \left(r - \frac{1}{2}v_u\right)du + \int_0^t \sqrt{v_u}dW_{1,u}.$$

Euler discretization of the process for $\ln S_t$ is thus

$$\begin{aligned}\ln S_{t+dt} &= \ln S_t + \left(r - \frac{1}{2}v_t\right)dt + \sqrt{v_t}\left(W_{1,t+dt} - W_{1,t}\right) \\ &= \ln S_t + \left(r - \frac{1}{2}v_t\right)dt + \sqrt{v_t dt}Z_s.\end{aligned} \tag{10}$$

Hence the Euler discretization of $S_t$ is

$$S_{t+dt} = S_t \exp\left(\left(r - \frac{1}{2}v_t\right)dt + \sqrt{v_t dt}Z_s\right).$$

Again, to avoid negative variances we must apply the full truncation or reflection scheme by replacing $v_t$ everywhere with $v_t^+$ or with $|v_t|$.

### 1.3.1 Process for $(S_t, v_t)$ or $(\ln S_t, v_t)$

Start with the initial values $S_0$ for the stock price and $v_0$ for the variance. Given a value for $v_t$ at time $t$, we first obtain $v_{t+dt}$ from

$$v_{t+dt} = v_t + \kappa(\theta - v_t)dt + \sigma\sqrt{v_t dt}Z_v$$

and we obtain $S_{t+dt}$ from

$$S_{t+dt} = S_t + rS_t dt + \sqrt{v_t dt}S_t Z_s$$

or from

$$S_{t+dt} = S_t \exp\left(\left(r - \frac{1}{2}v_t\right)dt + \sqrt{v_t dt}Z_s\right).$$

To generate $Z_v$ and $Z_s$ with correlation $\rho$, we first generate two independent standard normal variable $Z_1$ and $Z_2$, and we set $Z_v = Z_1$ and $Z_s = \rho Z_1 + \sqrt{1-\rho^2}Z_2$.

## 2 Milstein Scheme

This scheme is described in Glasserman [2] and in Kloeden and Platen [4] for general processes, and in Kahl and Jackel [3] for stochastic volatility models. The scheme works for SDEs for which the coefficients $\mu(S_t)$ and $\sigma(S_t)$ depend only on $S$, and do not depend on $t$ directly. Hence we assume that the stock price $S_t$ is driven by the SDE

$$\begin{aligned} dS_t &= \mu(S_t)\,dt + \sigma(S_t)\,dW_t \qquad (11) \\ &= \mu_t dt + \sigma_t dW_t. \end{aligned}$$

In integral form

$$S_{t+dt} = S_t + \int_t^{t+dt} \mu_s ds + \int_t^{t+dt} \sigma_s dW_s. \qquad (12)$$

The key to the Milstein scheme is that the accuracy of the discretization is increased by considering expansions of the coefficients $\mu_t = \mu(S_t)$ and $\sigma_t = \sigma(S_t)$ via Itō's lemma. This is sensible since the coefficients are functions of $S$. Indeed, we can apply Itō's Lemma to the functions $\mu_t$ and $\sigma_t$ as we would for any differentiable function of $S$. By Itō's lemma, then, the SDEs for the coefficients are

$$\begin{aligned} d\mu_t &= \left(\mu_t'\mu_t + \frac{1}{2}\mu_t''\sigma_t^2\right)dt + (\mu_t'\sigma_t)\,dW_t \\ d\sigma_t &= \left(\sigma_t'\mu_t + \frac{1}{2}\sigma_t''\sigma_t^2\right)dt + (\sigma_t'\sigma_t)\,dW_t \end{aligned}$$

where the prime refers to differentiation in $S$ and where the derivatives in $t$ are zero because we assume that $\mu_t$ and $\sigma_t$ have no direct dependence on $t$. The integral form of the coefficients at time $s$ (with $t < s < t + dt$)

$$\begin{aligned} \mu_s &= \mu_t + \int_t^s \left(\mu_u'\mu_u + \frac{1}{2}\mu_u''\sigma_u^2\right)du + \int_t^s (\mu_u'\sigma_u)\,dW_u \\ \sigma_s &= \sigma_t + \int_t^s \left(\sigma_u'\mu_u + \frac{1}{2}\sigma_u''\sigma_u^2\right)du + \int_t^s (\sigma_u'\sigma_u)\,dW_u. \end{aligned}$$

Substitute for $\mu_s$ and $\sigma_s$ in (12) to produce

$$\begin{aligned} S_{t+dt} &= S_t + \int_t^{t+dt} \left(\mu_t + \int_t^s \left(\mu_u'\mu_u + \frac{1}{2}\mu_u''\sigma_u^2\right)du + \int_t^s (\mu_u'\sigma_u)\,dW_u\right)ds \\ &\quad + \int_t^{t+dt} \left(\sigma_t + \int_t^s \left(\sigma_u'\mu_u + \frac{1}{2}\sigma_u''\sigma_u^2\right)du + \int_t^s (\sigma_u'\sigma_u)\,dW_u\right)dW_s \end{aligned}$$

The terms higher than order one are $dsdu = \mathcal{O}\left((dt)^2\right)$ and $dsdW_u = \mathcal{O}\left((dt)^{3/2}\right)$ and are ignored. The term involving $dW_u dW_s$ is retained since $dW_u dW_s =$

$\mathcal{O}(dt)$ is of order one. This leaves us with

$$S_{t+dt} = S_t + \mu_t \int_t^{t+dt} ds + \sigma_t \int_t^{t+dt} dW_s + \int_t^{t+dt} \int_t^s (\sigma'_u \sigma_u) \, dW_u dW_s. \quad (13)$$

Apply Euler discretization to the last term to obtain

$$\begin{aligned} \int_t^{t+dt} \int_t^s \sigma'_u \sigma_u dW_u dW_s &\approx \sigma'_t \sigma_t \int_t^{t+dt} \int_t^s dW_u dW_s \quad (14) \\ &= \sigma'_t \sigma_t \int_t^{t+dt} (W_s - W_t) \, dW_s \\ &= \sigma'_t \sigma_t \left( \int_t^{t+dt} W_s dW_s - W_t W_{t+dt} + W_t^2 \right) \end{aligned}$$

Now define $dY_t = W_t dW_t$. Using Itō's Lemma, it is easy to show[1] that $Y_t$ has solution $Y_t = \frac{1}{2}W_t^2 - \frac{1}{2}t$ so that

$$\int_t^{t+dt} W_s dW_s = Y_{t+dt} - Y_t = \frac{1}{2}W_{t+dt}^2 - \frac{1}{2}W_t^2 - \frac{1}{2}dt. \quad (15)$$

Substitute back into (14) to obtain

$$\begin{aligned} \int_t^{t+dt} \int_t^s \sigma'_u \sigma_u dW_u dW_s &\approx \frac{1}{2}\sigma'_u \sigma_u \left[ (W_{t+dt} - W_t)^2 - dt \right] \\ &= .\frac{1}{2}\sigma'_u \sigma_u \left[ (\Delta W_t)^2 - dt \right]. \end{aligned}$$

where $\Delta W_t = W_{t+dt} - W_t$, which is equal in distribution to $\sqrt{dt}Z$ with $Z$ distributed as standard normal. Combining Equations (13) and (15) the general form of Milstein discretization is therefore

$$S_{t+dt} = S_t + \mu_t dt + \sigma_t \sqrt{dt} Z + \frac{1}{2}\sigma'_t \sigma_t dt \left( Z^2 - 1 \right). \quad (16)$$

## 2.1 Milstein Scheme for the Black-Scholes Model

In the Black-Scholes model Equation (4) we have $\mu(S_t) = rS_t$ and $\sigma(S_t) = \sigma S_t$ so the Milstein scheme (16) is

$$S_{t+dt} = S_t + rS_t dt + \sigma S_t \sqrt{dt} Z + \frac{1}{2}\sigma^2 dt \left( Z^2 - 1 \right)$$

which adds the correction term $\frac{1}{2}\sigma^2 dt \left( Z^2 - 1 \right)$ to the Euler scheme in (5). In the Black-Scholes model for the log-stock price, Equation (6), we have $\mu(S_t) =$

[1] Indeed, $\frac{\partial Y}{\partial t} = -\frac{1}{2}$, $\frac{\partial Y}{\partial W} = W$, and $\frac{\partial^2 Y}{\partial W^2} = 1$, so that $dY_t = \left( -\frac{1}{2} + 0 + \frac{1}{2} \cdot 1 \cdot 1 \right) dt + (W_t \cdot 1) \, dW_t = W_t dW_t$.

$\left(r - \frac{1}{2}\sigma^2\right)$ and $\sigma(S_t) = \sigma$ so that $\mu_t' = \sigma_t' = 0$. The Milstein scheme (16) is therefore

$$\ln S_{t+dt} = \ln S_t + \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma\sqrt{dt}Z$$

which is identical to the Euler scheme in (7). Hence, while the Milstein scheme improves the discretization of $S_t$ in the Black-Scholes model, it does not improve the discretization of $\ln S_t$.

## 2.2 Milstein Scheme for the Heston Model

Recall that this model is given in Equation (8) as

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t}S_t dW_{1,t} \\ dv_t &= \kappa(\theta - v_t)\, dt + \sigma\sqrt{v_t}dW_{2,t} \end{aligned}$$

## 2.3 Process for $v_t$

The coefficients of the variance process are $\mu(v_t) = \kappa(\theta - v_t)$ and $\sigma(v_t) = \sigma\sqrt{v_t}$ so an application of Equation (16) for $v_t$ produces

$$v_{t+dt} = v_t + \kappa(\theta - v_t)\, dt + \sigma\sqrt{v_t dt}Z_v + \frac{1}{4}\sigma^2 dt\left(Z_v^2 - 1\right) \tag{17}$$

which can be written

$$v_{t+dt} = \left(\sqrt{v_t} + \frac{1}{2}\sigma\sqrt{dt}Z_v\right)^2 + \kappa(\theta - v_t)\, dt - \frac{1}{4}\sigma^2 dt.$$

This last equation is also Equation (2.18) of Gatheral [1]. Milstein discretization of the variance process produces far fewer negative values for the variance than Euler discretization. Nevertheless, the full truncation scheme or the reflection scheme must be applied to (17) as well.

### 2.3.1 Process for $S_t$ and $\ln S_t$

The coefficients of the stock price process are $\mu(S_t) = rS_t$ and $\sigma(S_t) = \sqrt{v_t}S_t$ so Equation (16) becomes

$$S_{t+dt} = S_t + rS_t dt + \sqrt{v_t dt}S_t Z_s + \frac{1}{4}S_t^2 dt\left(Z_s^2 - 1\right). \tag{18}$$

We can also discretize the log-stock process, which by Itō's lemma follows the process

$$d\ln S_t = \left(r - \frac{1}{2}v_t\right) dt + \sqrt{v_t}dW_{1,t}.$$

The coefficients are $\mu(S_t) = \left(r - \frac{1}{2}v_t\right)$ and $\sigma(S_t) = \sqrt{v_t}$ so that $\mu_t' = \sigma_t' = 0$. Since $v_t$ is known at time $t$, we can treat it as a constant in the coefficients. An application of (16) produces

$$\ln S_{t+dt} = \ln S_t + \left(r - \frac{1}{2}v_t\right)dt + \sqrt{v_t dt}dZ_s$$

which is identical to Equation (10). Hence, as in the Black-Scholes model, the discretization of $\ln S_t$ rather than $S_t$ means that there are no higher corrections to be brought to the Euler discretization. The discretization of the stock price is

$$S_{t+dt} = S_t \exp\left(\left(r - \frac{1}{2}v_t\right)dt + \sqrt{v_t dt}Z_s\right). \tag{19}$$

Again, it is necessary to apply the full truncation or reflections schemes in Equations (18) and (19).

### 2.4 Process for $(S_t, v_t)$ or $(\ln S_t, v_t)$

Given a value for $v_t$ at time $t$, we first update to $v_{t+dt}$ using (17)

$$v_{t+dt} = v_t + \kappa(\theta - v_t)dt + \sigma\sqrt{v_t dt}Z_v + \frac{1}{4}\sigma^2 dt\left(Z_v^2 - 1\right)$$

and we obtain $S_{t+dt}$ using

$$S_{t+dt} = S_t + rS_t dt + \sqrt{v_t dt}S_t Z_s + \frac{1}{4}S_t^2 dt\left(Z_s^2 - 1\right).$$

or from

$$S_{t+dt} = S_t \exp\left(\left(r - \frac{1}{2}v_t\right)dt + \sqrt{v_t dt}Z_s\right).$$

To generate $Z_v$ and $Z_s$ with correlation $\rho$, we first generate two independent standard normal variable $Z_1$ and $Z_2$, and we set $Z_v = Z_1$ and $Z_s = \rho Z_1 + \sqrt{1-\rho^2}Z_2$.

## References


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