Payoff Function Decomposition

by Fabrice Douglas Rouah
www.FRouah.com
www.Volopta.com

The following identify is often used in option pricing. For any twice-differentiable function $f(S)$ defined on $\mathbb{R}$ we can write

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) + \\ &\quad \int_{-\infty}^{\kappa} f''(K)(K-S)^+ dK + \int_{\kappa}^{\infty} f''(K)(S-K)^+ dK \end{aligned} \tag{1}$$

for some threshold κ. I have seen three proofs of this.

1. Uses the Dirac delta function. Found in papers by Carr and Madan.
2. Uses the Fundamental Theorem of Calculus. Found in a paper by John Crosby at Glasgow University.
3. Uses the Taylor Series expansion of $f(S)$. Found in a paper by Attilio Meucci.

In this Note these three proofs are each presented in detail. We first present some functions that will be needed.

1 Dirac Delta and Other Functions

Definition of the Dirac delta function $\delta(K)$

$$\delta(K) = \begin{cases} \infty \text{ for } K = 0 \\ 0 \text{ elsewhere} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(K) dK = 1.$$

The sifting property of the Dirac delta function is

$$f(a) = \int_{a-\varepsilon}^{a+\varepsilon} f(K)\delta(a-K)\, dK$$

for $\varepsilon > 0$. The Heaviside function $H(K)$ is defined as

$$H(K) = \begin{cases} 1 \text{ for } K > 0 \\ 0 \text{ for } K < 0 \end{cases} = \mathbf{1}_{(K>0)}.$$

The derivative of $H(K)$ is easy to check

$$\frac{d}{dK} H(K) = \delta(K).$$

The derivative of $(K)^+ = \max(0, K)$ is also easy to check

$$\frac{d}{dK}(K)^+ = H(K).$$

The definitions and derivatives needed for the proofs in this Note are presented in the following table.

For $(S-K)$	For $(K-S)$
$\delta(S-K) = \begin{cases} \infty \text{ for } K = S \\ 0 \text{ elsewhere} \end{cases}$	$\delta(K-S) = \begin{cases} \infty \text{ for } S = K \\ 0 \text{ elsewhere} \end{cases}$
$H(S-K) = \mathbf{1}_{(S>K)}$	$H(K-S) = \mathbf{1}_{(K>S)}$
$\frac{d}{dK}H(S-K) = \delta(S-K)$	$\frac{d}{dK}H(K-S) = \delta(K-S)$
$\frac{d}{dK}(S-K)^+ = -\mathbf{1}_{(S>K)}$	$\frac{d}{dK}(K-S)^+ = \mathbf{1}_{(K>S)}$

2 Proof Using the Dirac Delta Function

Carr and Madan consider $S > 0$ in equation (1), since they use it for an underlying price S.

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) + \\ &\quad \int_0^\kappa f''(K)(K-S)^+ dK + \int_\kappa^\infty f''(K)(S-K)^+ dK. \end{aligned} \tag{2}$$

Apply the sifting property of the Dirac delta function to $f(S)$ and choose some threshold $\kappa > 0$

$$\begin{aligned} f(S) &= \int_0^\infty f(K)\delta(S-K)dK \\ &= \int_0^\kappa f(K)\delta(S-K)dK + \int_\kappa^\infty f(K)\delta(S-K)dK \\ &= \int_0^\kappa f(K)\delta(K-S)dK + \int_\kappa^\infty f(K)\delta(S-K)dK \\ &= I_1 + I_2. \end{aligned} \tag{3}$$

2.1 First Integration by Parts

Apply integration by parts to each of the two integrals I_1 and I_2.

2.1.1 Evaluating I_1

For I_1 choose $u = f(K), v' = \delta(K - S)$. Hence $u' = f'(K), v = \mathbf{1}_{(K>S)}$.

$$
\begin{aligned}
I_1 &= \int_0^{\kappa} f(K)\delta(K - S)dK && (4)\\
&= f(K)\mathbf{1}_{(K>S)}\big|_0^{\kappa} - \int_0^{\kappa} f'(K)\mathbf{1}_{(K>S)}dK && (5)\\
&= f(\kappa)\mathbf{1}_{(\kappa>S)} - \int_0^{\kappa} f'(K)\mathbf{1}_{(K>S)}dK.
\end{aligned}
$$

2.1.2 Evaluating I_2

For I_2 choose $u = f(K), v' = \delta(S - K)$ so that $u' = f'(K), v = -\mathbf{1}_{(K>S)}$.

$$
\begin{aligned}
I_2 &= \int_{\kappa}^{\infty} f(K)\delta(S - K)dK && (6)\\
&= -f(K)\mathbf{1}_{(S>K)}\big|_{\kappa}^{\infty} + \int_{\kappa}^{\infty} f'(K)\mathbf{1}_{(K<S)}dK && (7)\\
&= f(\kappa)\mathbf{1}_{(S>\kappa)} + \int_{\kappa}^{\infty} f'(K)\mathbf{1}_{(K<S)}dK.
\end{aligned}
$$

Substitute I_1 and I_2 from equations (4) and (6) into equation (3) to obtain

$$
\begin{aligned}
f(S) &= f(\kappa) - \int_0^{\kappa} f'(K)\mathbf{1}_{(K>S)}dK + \int_{\kappa}^{\infty} f'(K)\mathbf{1}_{(K<S)}dK && (8)\\
&= f(\kappa) - I_3 + I_4.
\end{aligned}
$$

2.2 Second Integration by Parts

Apply integration by parts again, to each of the two integrals I_3 and I_4.

2.2.1 Evaluating I_3

For I_3 choose $u = f'(K), v = \mathbf{1}_{(K>S)}, u' = f''(K), v' = (K - S)^+$. Hence

$$
\begin{aligned}
I_3 &= \int_0^{\kappa} f'(K)\mathbf{1}_{(K>S)}dK && (9)\\
&= f'(K)(K - S)^+\big|_0^{\kappa} - \int_0^{\kappa} f''(K)(K - S)^+dK && (10)\\
&= f'(\kappa)(\kappa - S)^+ - \int_0^{\kappa} f''(K)(K - S)^+dK.
\end{aligned}
$$

2.2.2 Evaluating I_4

Finally, for I_4 choose $u = f'(K), v = \mathbf{1}_{(S>K)}, u' = f''(K), v' = -(S-K)^+$.

$$\begin{aligned} I_4 &= \int_\kappa^\infty f'(K)\mathbf{1}_{(K<S)}dK \quad (11) \\ &= -f'(K)(S-K)^+\big|_\kappa^\infty + \int_\kappa^\infty f''(K)(S-K)^+dK \quad (12) \\ &= -f'(\kappa)(S-\kappa)^+ + \int_\kappa^\infty f''(K)(S-K)^+dK. \end{aligned}$$

2.3 Obtaining the Payoff Decomposition

Substitute I_3 and I_4 from equations (9) and (11) into equation (8) to btain

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)\left[(S-\kappa)^+ - (\kappa-S)^+\right] \quad (13) \\ &\quad + \int_0^\kappa f''(K)(K-S)^+dK + \int_\kappa^\infty f''(K)(S-K)^+dK. \end{aligned}$$

Equivalently, since $[(S-\kappa)^+ - (\kappa-S)^+] = S-\kappa$ (which can be verified by considering the cases $S>\kappa$ and $S<\kappa$ separately), equation (13) can be written

$$\begin{aligned} f(S) &= f(\kappa) + f'(\kappa)(S-\kappa) + \\ &\quad \int_0^\kappa f''(K)(K-S)^+dK + \int_\kappa^\infty f''(K)(S-K)^+ \end{aligned}$$

which is equation (2).

3 Proof Using the FTC

This proof uses the Fundamental Theorem of Calculus (FTC)

$$f(S) - f(\kappa) = \int_\kappa^S f'(u)du.$$

Hence we can write $f(S)$ as

$$f(S) = f(\kappa) + \int_\kappa^S f'(u)du. \quad (14)$$

We will focus on the integral $\int_\kappa^S f'(u)du$ in equation (14). Note that the integral can be broken into two parts: when $S>\kappa$ and when $S<\kappa$

$$\begin{aligned} \int_\kappa^S f'(u)du &= \mathbf{1}_{(S>\kappa)}\int_\kappa^S f'(u)du + \mathbf{1}_{(S<\kappa)}\int_\kappa^S f'(u)du \quad (15) \\ &= \mathbf{1}_{(S>\kappa)}\int_\kappa^S f'(u)du - \mathbf{1}_{(S<\kappa)}\int_S^\kappa f'(u)du. \end{aligned}$$

In equation (15) apply the FTC to the first integral so that $f'(u) - f'(\kappa) = \int_\kappa^u f''(v)dv$ and to the second integral so that $f'(u) - f'(\kappa) = -\int_u^\kappa f''(v)dv$

$$\begin{aligned}\int_\kappa^S f'(u)du &= \mathbf{1}_{(S>\kappa)} \int_\kappa^S \left[f'(\kappa) + \int_\kappa^u f''(v)dv\right] du - \\ &\quad \mathbf{1}_{(S<\kappa)} \int_S^\kappa \left[f'(\kappa) - \int_u^\kappa f''(v)dv\right] du.\end{aligned} \tag{16}$$

Re-arrange the integrals to obtain

$$\begin{aligned}\int_\kappa^S f'(u)du &= \mathbf{1}_{(S>\kappa)} \int_\kappa^S f'(\kappa)du - \mathbf{1}_{(S<\kappa)} \int_S^\kappa f'(\kappa)du + \\ &\quad \mathbf{1}_{(S>\kappa)} \int_\kappa^S \int_\kappa^u f''(v)dvdu + \mathbf{1}_{(S<\kappa)} \int_S^\kappa \int_u^\kappa f''(v)dvdu.\end{aligned} \tag{17}$$

Consider each of these four integrals in equation (17) in pairs.

3.1 First Pair of Integrals

The first two integrals in equation (17) are

$$\begin{aligned}&\mathbf{1}_{(S>\kappa)} \int_\kappa^S f'(\kappa)du - \mathbf{1}_{(S<\kappa)} \int_S^\kappa f'(\kappa)du \\ &= \mathbf{1}_{(S>\kappa)} \int_\kappa^S f'(\kappa)du + \mathbf{1}_{(S<\kappa)} \int_\kappa^S f'(\kappa)du \\ &= \int_\kappa^S f'(\kappa)du = f'(\kappa) \int_\kappa^S du = f'(\kappa)(S-\kappa).\end{aligned} \tag{18}$$

3.2 Second Pair of Integrals

The last two integrals in equation (1) can be evaluated using Fubini's theorem – changing the order of integration, making sure to preserve the same domain of integration. The integral

$$\int_\kappa^S \int_\kappa^u f''(v)dvdu \tag{19}$$

has domain of integration $\left\{\begin{smallmatrix}\kappa<v<u \\ \kappa<u<S\end{smallmatrix}\right\}$, which is equivalent to $\left\{\begin{smallmatrix}v<u<S \\ \kappa<v<S\end{smallmatrix}\right\}$. Hence we can write the integral (19) as

$$\int_\kappa^S \int_\kappa^u f''(v)dvdu = \int_\kappa^S \int_v^S f''(v)dudv = \int_\kappa^S f''(v)(S-v)dv.$$

Similarly, the integral

$$\int_S^\kappa \int_u^\kappa f''(v)dvdu \tag{20}$$

has domain of integration $\left\{ {u<v<\kappa \atop S<u<\kappa} \right\}$, which is equivalent to $\left\{ {S<u<v \atop S<v<\kappa} \right\}$. Hence we can write the integral (20) as

$$\int_S^\kappa \int_u^\kappa f''(v)dvdu = \int_S^\kappa \int_S^v f''(v)dudv = \int_S^\kappa f''(v)(v-S)dv.$$

3.3 Recombing the Integrals

We put these two integrals back into equation (17), and substitute into equation (14). We obtain

$$f(S) = f(\kappa) + \mathbf{1}_{(S>\kappa)} \int_\kappa^S f''(v)(S-v)dv + \mathbf{1}_{(S<\kappa)} \int_S^\kappa f''(v)(v-S)dv. \quad (21)$$

Since $\mathbf{1}_{(S>\kappa)} = 1$ only when $S > \kappa$, and since $\mathbf{1}_{(S<\kappa)} = 1$ only when $S < \kappa$, these can be replaced by the $()^+$ function on the linear parts of each integrand in equation (21). Hence we have the desired result

$$f(S) = f(\kappa) + f'(\kappa)(S-\kappa) + \int_\kappa^\infty f''(v)(S-v)^+dv + \int_0^\kappa f''(v)(v-S)^+dv$$

which, again, is equation (1).

4 Proof Using Taylor Series

Define the function $g(x)$ as

$$g(x) = f(k)+f'(k)(x-k)+\int_k^\infty f''(u)(x-u)^+du+\int_{-\infty}^k f''(u)(u-x)^+du. \quad (22)$$

The proof involves demonstrating that $g(x) = f(x)$. This is done by showing that $g(k) = f(k)$ and that the derivatives of g and f evaluated at any point x are the same, i.e. $g^{(n)}(x) = f^{(n)}(x)$. The Taylor series expansion of f and g will therefore be identical, and consequently, so will f and g themselves, at all values of x. First note that for $x = k$, equation (22) becomes

$$g(k) = f(k)+f'(k)(k-k)+\int_k^\infty f''(u)(k-u)^+du+\int_{-\infty}^k f''(u)(u-k)^+du. \quad (23)$$

In equation (23), the first integral is zero since $(k-u)^+ = 0$ for $u > k$, and the second integral is zero since $(u-k)^+ = 0$ for $u < k$. Hence $g(k) = f(k)$.

4.1 First Derivative

Take the first derivative of $g(x)$ in equation (22)

$$\begin{aligned} g'(x) &= f'(k) + \int_k^\infty f''(u)\frac{d}{dx}(x-u)^+du + \int_{-\infty}^k f''(u)\frac{d}{dx}(u-x)^+du \quad (24) \\ &= f'(k) + \int_k^\infty f''(u)H(x-u)du - \int_{-\infty}^k f''(u)H(u-x)du. \end{aligned}$$

Consider the second line of equation (24). In the first integral, $H(x-u)=1$ only when $u<x$. Hence the upper limit can be replaced with x. Moreover, $H(x-u)=1$ only when $x>k$, in which case the integral is $\int_k^x f''(u)du$. This can be accomplished by including the term $H(x-k)$ outside the integral. Similarly, for the second integral, $H(u-x)=1$ only when $u>x$ so the lower limit can be replaced with x, and $H(u-x)=1$ only when $x<k$, so we put $H(k-x)$ outside the integral. This means that equation () can be written as

$$g'(x) = f'(k) + H(x-k)\int_k^x f''(u)du - H(k-x)\int_x^k f''(u)du. \qquad (25)$$

To evaluate equation (25) we consider the cases $x>k$ and $x<k$ separately.

4.1.1 Case 1

For $x>k$ we have $H(k-x)=0$ and

$$H(x-k)\int_k^x f''(u)du = \int_k^x f''(u)du = f'(x) - f'(k)$$

Hence equation (25) becomes

$$g'(x) = f'(k) + [f'(x) - f'(k)] - 0 = f'(x).$$

4.1.2 Case 2

For $x<k$ we have $H(x-k)=0$ and

$$H(k-x)\int_x^k f''(u)du = f'(k) - f(x).$$

Hence equation (25) becomes

$$g'(x) = f'(k) + 0 - [f'(k) - f'(x)] = f'(x).$$

So for both cases we have $g'(x) = f'(x)$.

4.2 Second Derivative

Take the second derivative of $g(x)$ in equation (22)

$$\begin{aligned} g''(x) &= \int_k^\infty f''(u)\frac{d^2}{dx^2}(x-u)^+du + \int_{-\infty}^k f''(u)\frac{d^2}{dx^2}(u-x)^+du \qquad (26) \\ &= \int_k^\infty f''(u)\delta(x-u)du + \int_{-\infty}^k f''(u)\delta(u-x)du. \end{aligned}$$

Consider the second line of equation (26). The sifting property of the Dirac delta function implies that the first integral is $f''(x)$, but only when $x>k$, since this ensures that the domain of integral is sufficiently wide for the sifting property to hold. Otherwise, the integral is zero. This can be accomplished by multiplying the integral by $H(x-k)$. Similarly, the second integral is $f''(x)$ only when $x < k$, so we multiply the integral by $H(k-x)$. This implies that we can write $g''(x)$ as

$$\begin{aligned} g''(x) &= H(x-k)\int_k^\infty f''(u)\delta(x-u)du + H(k-x)\int_{-\infty}^k f''(u)\delta(u-x)du \\ &= H(x-k)f''(x) + H(k-x)f''(x) \\ &= f''(x) \end{aligned}$$

where the second line follows by the sifting property of the Dirac delta function.

4.3 Higher Order Derivatives

The nth derivative of $g(x)$ is

$$\begin{aligned} \frac{d^n}{dx^n}g(x) &= \frac{d^n}{dx^n}\left[f(k) + f'(k)(x-k) + \int_k^\infty f''(u)(x-u)^+du + \int_{-\infty}^k f''(u)(u-x)^+du\right] \\ &= \frac{d^{n-2}}{dx^{n-2}}\left[\int_k^\infty f''(u)\frac{d^2}{dx^2}(x-u)^+du + \int_{-\infty}^k f''(u)\frac{d^2}{dx^2}(u-x)^+du\right]. \end{aligned}$$

From equation (26) in the preceding section we know that the term inside the square brackets is simply $\frac{d^2}{dx^2}f(x)$. Hence we can write

$$\frac{d^n}{dx^n}g(x) = \frac{d^{n-2}}{dx^{n-2}}\left[\frac{d^2}{dx^2}f(x)\right] = \frac{d^n}{dx^n}f(x).$$

Apply the Taylor theorem to $g(x)$

$$\begin{aligned} g(x) &= g(k) + \sum_{n=1}^\infty \frac{1}{n!}g^{(n)}(x-k)^n \\ &= f(k) + \sum_{n=1}^\infty \frac{1}{n!}f^{(n)}(x-k)^n \\ &= f(x). \end{aligned}$$

Since the functions f and g are identical, equation (22) becomes equation (1) and the proof is complete.