The Feynman-Kac Theorem

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In this Note we illustrate the Feynman-Kac theorem in one dimension, and in multiple dimensions. We illustrate the use of the theorem using the Black-Scholes and Heston models. The Feynman-Kac theorem is explained in detail in textbooks such as the one by Klebaner [2].

1 The Theorem in One Dimension

Suppose that x_t follows the stochastic process

$$dx_t = \mu(x_t, t)dt + \sigma(x_t, t)\, dW_t^{\mathbb{Q}} \tag{1}$$

where $W_t^{\mathbb{Q}}$ is Brownian motion under the measure $\mathbb{Q}$. Let $V(x_t, t)$ be a differentiable function of x_t and t and suppose that $V(x_t, t)$ follows the partial differential equation (PDE) given by

$$\frac{\partial V}{\partial t} + \mu(x_t, t)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma(x_t, t)^2 \frac{\partial^2 V}{\partial x^2} - r(x_t, t)V(x_t, t) = 0$$

and with boundary condition $V(X_T, T)$. The theorem asserts that $V(x_t, t)$ has the solution

$$V(x_t, t) = E^{\mathbb{Q}}\left[e^{-\int_t^T r(x_u, u)du} V(X_T, T) \middle| \mathcal{F}_t \right]. \tag{2}$$

Note that the expectation is taken under the measure $\mathbb{Q}$ that makes the stochastic term in Equation (1) Brownian motion. The generator of the process in (1) is defined as the operator

$$\mathcal{A} = \mu(x_t, t)\frac{\partial}{\partial x} + \frac{1}{2}\sigma(x_t, t)^2 \frac{\partial}{\partial x^2} \tag{3}$$

so the PDE in $V(x_t, t)$ is sometimes written

$$\frac{\partial V}{\partial t} + \mathcal{A}V(x_t, t) - r(x_t, t)V(x_t, t) = 0. \tag{4}$$

The Feynman-Kac theorem can be used in both directions. That is,

1. If we know that x_t follows the process in Equation (1) and we are given a function $V(x_t, t)$ with boundary condition $V(X_T, T)$, then we can always obtain the solution for $V(x_t, t)$ as Equation (2).

2. If we know that the solution to $V(x_t, t)$ is given by Equation (2) and that x_t follows the process in (1), then we are assured that $V(x_t, t)$ satisfies the PDE in Equation (4).

1.1 Example Using the Black-Scholes Model

Let the stock price S_t be driven by the process

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

The risk-neutral process is

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}} \tag{5}$$

where $W_t^{\mathbb{Q}} = \frac{\mu - r}{\sigma} t + W_t$. A derivative written on the stock follows the Black-Scholes PDE

$$\frac{\partial V}{dt} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{6}$$

The generator of the process given by Equation (5) is

$$\mathcal{A} = rS\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial}{\partial S^2}$$

so the Black-Scholes PDE in Equation (6) can also be written exactly as in (4) by substituting $r(x_u, u) = r$, a constant, in Equation (4). By the Feynman-Kac Theorem, the time-t value a the derivative with payoff $V(S_T, T)$ is the solution in Equation (2)

$$V(S_t, t) = e^{-r(T-t)} E^{\mathbb{Q}} \left[V(S_T, T) | \mathcal{F}_t \right]. \tag{7}$$

1.1.1 European Call

The time-t value $V(S_t, t)$ of a European call option written on S_t with strike price K has the payoff $V(S_T, T) = \max(S_T - K, 0)$. We substitute this payoff in the Feynman-Kac formula in Equation (7). Hence, the time-t value of a European call is

$$V(S_t, t) = e^{-r(T-t)} E^{\mathbb{Q}} \left[\max(S_T - K, 0) | \mathcal{F}_t \right]. \tag{8}$$

The solution is the Black-Scholes time-t price of a European call

$$V(S_t, t) = S_t \Phi(d_1) - e^{-r(T-t)} K \Phi(d_2) \tag{9}$$

where $d_1 = \frac{\ln S/K + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$, and $\Phi(y)$ is the standard normal cdf. See the Note on www.FRouah.com for several ways in which the Black-Scholes European call price (9) can be derived.

1.1.2 Binary Options

The cash-or-nothing call pays a fixed amount X if $S_T > K$ and nothing otherwise. We can price this option using the Feynman-Kac theorem under the Black-Scholes dynamics. The boundary condition is $V(S_T, T) = X\mathbf{1}_{S_T > K}$ so the time-t value of the cash-or-nothing call is, by Equation (7)

$$\begin{aligned} V(S_t, t) &= e^{-r(T-t)} E^{\mathbb{Q}} \left[X\mathbf{1}_{S_T > K} | \mathcal{F}_t \right] \\ &= Xe^{-r(T-t)} \mathbb{Q}(S_T > K). \end{aligned}$$

It can be shown that S_T follows the lognormal distribution and that the probability $\mathbb{Q}(S_T > K) = \int_K^\infty dF(S_T)$ can be expressed in terms of d_2 as $\mathbb{Q}(S_T > K) = \Phi(d_2)$. Hence the value of the cash-or-nothing call is

$$V(S_t, t) = Xe^{-r(T-t)}\Phi(d_2).$$

Similarly, the asset-or-nothing call pays S_T if $S_T > K$ and nothing otherwise. The boundary condition is therefore $V(S_T, T) = S_T \mathbf{1}_{S_T > K}$ and the time-t value of the asset-or-nothing call is, by Equation (7)

$$\begin{aligned} V(S_t, t) &= e^{-r(T-t)} E^{\mathbb{Q}} \left[S_T \mathbf{1}_{S_T > K} \middle| \mathcal{F}_t \right] \\ &= e^{-r(T-t)} E^{\mathbb{Q}} \left[S_T \middle| S_T > K, \mathcal{F}_t \right]. \end{aligned}$$

It can be shown that the conditional expected value $E^{\mathbb{Q}}[S_T | S_T > K, \mathcal{F}_t] = \int_K^\infty S_T dF(S_T)$ can be expressed in terms of d_1 as $S_t e^{r(t-t)}\Phi(d_1)$. Hence, the value of the asset-or-nothing call is

$$V(S_t, t) = S_t \Phi(d_1).$$

1.1.3 The European Call Replicated by Binary Options

The binary options in the preceding section show that the European call can be replicated by

1. A long position in an asset-or-nothing call with strike K, and
2. A short position in a cash-or-nothing call that pays K and with strike K.

This can also be seen by writing the European call formula in Equation (8) as

$$\begin{aligned} V(S_t, t) &= e^{-r(T-t)} E^{\mathbb{Q}} \left[\max(S_T - K, 0) \middle| \mathcal{F}_t \right] \\ &= e^{-r(T-t)} E^{\mathbb{Q}} \left[S_T \mathbf{1}_{S_T > K} \middle| \mathcal{F}_t \right] - e^{-r(T-t)} E^{\mathbb{Q}} \left[K \mathbf{1}_{S_T > K} \middle| \mathcal{F}_t \right]. \end{aligned}$$

See the Note on www.FRouah.com on the Black-Scholes formula for an explanation of how the lognormal probability $\mathbb{Q}(S_T > K)$ and conditional expectation $E^{\mathbb{Q}}[S_T | S_T > K]$ are derived.

2 Multi-Dimensional Version of the Theorem

Suppose that $\mathbf{x}_t$ follows the stochastic process in n dimensions

$$d\mathbf{x}_t = \boldsymbol{\mu}(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}(\mathbf{x}_t, t)d\mathbf{W}_t^{\mathbb{Q}}$$

where $\mathbf{x}_t$ and $\boldsymbol{\mu}(\mathbf{x}_t, t)$ are each vectors of dimension n, $\mathbf{W}_t^{\mathbb{Q}}$ is a vector of dimension m of $\mathbb{Q}$-Brownian motion, and $\boldsymbol{\sigma}(\mathbf{x}_t, t)$ is a matrix of size $n \times m$. In other words

$$d\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} \mu_1(\mathbf{x}_t,t) \\ \vdots \\ \mu_n(\mathbf{x}_t,t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(\mathbf{x}_t,t) & \cdots & \sigma_{1m}(\mathbf{x}_t,t) \\ \vdots & \ddots & \vdots \\ \sigma_{n1}(\mathbf{x}_t,t) & \cdots & \sigma_{nm}(\mathbf{x}_t,t) \end{pmatrix} \begin{pmatrix} dW_1^{\mathbb{Q}}(t) \\ \vdots \\ dW_m^{\mathbb{Q}}(t) \end{pmatrix}.$$

The generator of the process is

$$\mathcal{A} = \sum_{i=1}^{n} \mu_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \tag{10}$$

where for notational convenience $\mu_i = \mu_i(\mathbf{x}_t, t)$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}_t, t)$, and $\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)_{ij}$ is element (i, j) of the matrix $\boldsymbol{\sigma}\boldsymbol{\sigma}^T$ of size $(n \times n)$. The theorem states that the partial differential equation (PDE) in $V(\mathbf{x}_t, t)$ given by

$$\frac{\partial V}{\partial t} + \mathcal{A}V(\mathbf{x}_t, t) - r(\mathbf{x}_t, t)V(\mathbf{x}_t, t) = 0 \tag{11}$$

and with boundary condition $V(\mathbf{X}_T, T)$ has solution

$$V(\mathbf{x}_t, t) = E^{\mathbb{Q}}\left[e^{-\int_t^T r(\mathbf{x}_u, u)du} V(\mathbf{X}_T, T) \middle| \mathcal{F}_t \right]. \tag{12}$$

2.1 Example Using the Heston Model

2.1.1 The Heston Model as a Bivariate Process and its Generator

In Heston's model [1], Itō's lemma can be applied to the processes for the stock price S_t and variance v_t to produce the processes for the logarithm of the stock price $x_t = \ln S_t$ and the variance v_t. Under the EMM $\mathbb{Q}$, these are

$$\begin{aligned} dx &= \left(r - \frac{1}{2}v\right) dt + \sqrt{v} dW_1^{\mathbb{Q}} \\ dv &= \kappa\left(\theta - v\right) dt + \sigma\sqrt{v} dW_2^{\mathbb{Q}}. \end{aligned} \tag{13}$$

The process for $\mathbf{x} = (x, v)$ can be written in terms of two independent Brownian motions Z_1 and Z_2 as[1]

$$d\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} r - \frac{1}{2}v \\ \kappa\left(\theta - v\right) \end{pmatrix} dt + \begin{pmatrix} \sqrt{v} & 0 \\ \sigma\rho\sqrt{v} & \sigma\sqrt{v\left(1 - \rho^2\right)} \end{pmatrix} \begin{pmatrix} dZ_1 \\ dZ_2 \end{pmatrix}. \tag{14}$$

[1] The Browian motions W_1 and W_2 have correlation ρ and can be expressed in terms of two independent Brownian motions Z_1 and Z_2 as $dW_1 = dZ_1$ and $dW_2 = \rho dZ_1 + \sqrt{1 - \rho^2} dZ_2$.

To obtain the generator in Equation (10), we need the following matrix from (14)

$$\begin{aligned}\boldsymbol{\sigma}\boldsymbol{\sigma}^T &= \begin{pmatrix} \sqrt{v} & 0 \\ \sigma\rho\sqrt{v} & \sigma\sqrt{v\left(1-\rho^2\right)} \end{pmatrix}\begin{pmatrix} \sqrt{v} & \sigma\rho\sqrt{v} \\ 0 & \sigma\sqrt{v(1-\rho)^2} \end{pmatrix} \\ &= \begin{pmatrix} v & \sigma\rho v \\ \sigma\rho v & \sigma^2 v \end{pmatrix}.\end{aligned}$$

The generator in Equation (10) is therefore

$$\mathcal{A} = \left(r - \frac{1}{2}v\right)\frac{\partial}{\partial x} + \kappa\left(\theta - v\right)\frac{\partial}{\partial v} + \frac{1}{2}\left[v\frac{\partial^2}{\partial x^2} + \sigma^2 v\frac{\partial^2}{\partial v^2} + 2\sigma\rho v\frac{\partial^2}{\partial x\partial v}\right]. \quad (15)$$

The PDE in Equation (11) for $V = V(x, v, t)$ becomes

$$\begin{aligned}&\frac{\partial V}{\partial t} + \left(r - \frac{1}{2}v\right)\frac{\partial V}{\partial x} + \kappa\left(\theta - v\right)\frac{\partial V}{\partial v} \quad (16)\\ &+\frac{1}{2}v\frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} + \sigma\rho v\frac{\partial^2 V}{\partial x\partial v} - rV = 0\end{aligned}$$

which is Equation (6) of Heston [1] with $r(x_t, t) = r$ (a constant), and with $\lambda(x, v, t) = 0$.

2.1.2 The Call Option Value

In a general setting of non-constant interest rates r_u the value of a European call option is

$$\begin{aligned}C(S_t, t) &= E^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\max(S_T - K, 0)\right] \quad (17)\\ &= E^{\mathbb{Q}}\left[\frac{B_t}{B_T}S_T\mathbf{1}_{S_T>K}\right] - KE^{\mathbb{Q}}\left[\frac{B_t}{B_T}\mathbf{1}_{S_T>K}\right].\end{aligned}$$

where $B_t = \exp\left(\int_0^t r_u\right)$ is time-t value of the money-market account. Both time-t expectations $E^{\mathbb{Q}}[\cdot]$ are conditional on the time-t information set (x_t, v_t, t).

2.1.3 Change of Measure

The objective is to end up with expectations in Equation (17) in which only the indicator function $\mathbf{1}_{S_T>K}$ remains in each. This is done by changing the numeraires in the expectations and using these new numeraires to produce two Radon-Nikodym derivatives. These Radon-Nikodym derivatives will allow the measure $\mathbb{Q}$ to be changed to new measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$. That way, the expectations can be expressed as probabilities $\mathbb{Q}_j\left(S_T > K\right)$, albeit under different measures. We then express $\mathbb{Q}_j\left(S_T > K\right)$ as $\mathbb{Q}_j\left(x_T > \ln K\right)$ where $x_T = \ln S_T$ and apply the inversion theorem, according to which we can recover the probability from the characteristic function $\varphi_j(u; x_t, v_t, t)$ for x_T as

$$\mathbb{Q}_j(x_T > \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-iu\ln K}\varphi_j(u; x_t, v_t, t)}{iu}\right) du. \tag{18}$$

For the first expectation in the second line of Equation (17), we change the risk-neutral measure from $\mathbb{Q}$ to $\mathbb{Q}_1$ by using the Radon-Nikodym derivative $\mathbb{Z}_t$ defined as

$$\mathbb{Z}_t = \frac{d\mathbb{Q}}{d\mathbb{Q}_1} = \frac{S_t/S_T}{B_t/B_T}.$$

Hence the first expectation in Equation (17) can be written as

$$\begin{aligned} E^{\mathbb{Q}}\left[\frac{B_t}{B_T}S_T\mathbf{1}_{S_T>K}\right] &= E^{\mathbb{Q}_1}\left[\frac{B_t}{B_T}S_T\mathbf{1}_{S_T>K}\mathbb{Z}_t\right] \\ &= E^{\mathbb{Q}_1}\left[S_t\mathbf{1}_{S_T>K}\right] \\ &= S_t\mathbb{Q}_1\left(S_T > K\right). \end{aligned} \tag{19}$$

For the second expectation we use the Radon-Nikodym derivative we use the price of a zero-coupon bond $P_{t,T}$ as the numeraire, where

$$P_{t,T} = E\left[e^{-\int_t^T r_u du}\right]. \tag{20}$$

We change the risk-neutral measure from $\mathbb{Q}$ to $\mathbb{Q}_2$ by using the Radon-Nikodym derivative $\mathbb{Y}_t$ defined as

$$\mathbb{Y}_t = \frac{d\mathbb{Q}}{d\mathbb{Q}_2} = \frac{P_{t,T}/P_{T,T}}{B_t/B_T}.$$

Hence the second expectation in Equation (17) can be written as

$$\begin{aligned} E^{\mathbb{Q}}\left[\frac{B_t}{B_T}\mathbf{1}_{S_T>K}\right] &= E^{\mathbb{Q}_2}\left[\frac{B_t}{B_T}\mathbf{1}_{S_T>K}\mathbb{Y}_t\right] \\ &= E^{\mathbb{Q}_2}\left[P_{t,T}\mathbf{1}_{S_T>K}\right] \\ &= P_{t,T}\mathbb{Q}_2\left(S_T > K\right). \end{aligned} \tag{21}$$

Substituting the expectations in Equations (19) and (21) into the valuation formula (17) produces

$$C(S_t, t) = S_t\mathbb{Q}_1\left(x_T > \ln K\right) - P_{t,T}K\mathbb{Q}_2\left(x_T > \ln K\right)$$

which is the price of a European call in the Heston model.

2.1.4 The Characteristic Function and the Feynman-Kac Theorem

The point of this example is that there is a link between the characteristic functions φ_j $(j = 1, 2)$ and the Feynman-Kac theorem. In the Heston [1] model, interest rates are constant. From Equation (12), when $r(\mathbf{x}_u, u) = r$, a constant, the value $V(\mathbf{x}_t, t)$ becomes

$$V(\mathbf{x}_t, t) = e^{-r(T-t)} E^{\mathbb{Q}} \left[V(\mathbf{X}_T, T) \right].$$

Set $\mathbf{X}_T = \ln S_T = x_T$ and consider the functions $f_j(x_T, T) = E^{\mathbb{Q}_j} \left[e^{iux_T} \right]$. By the Feynman-Kac theorem, we know that this is the solution to a function $f(x_t, t)$ that follows the PDE given in Equation (16), written here in terms of f_j

$$\begin{aligned} &\frac{\partial f_j}{\partial t} + \left(r - \frac{1}{2} v \right) \frac{\partial f_j}{\partial x} + \kappa \left(\theta - v \right) \frac{\partial f_j}{\partial v} \\ &+ \frac{1}{2} v \frac{\partial^2 f_j}{\partial x^2} + \frac{1}{2} \sigma^2 v \frac{\partial^2 f_j}{\partial v^2} + \sigma \rho v \frac{\partial^2 f_j}{\partial x \partial v} - r f_j = 0 \end{aligned}$$

and that has boundary condition $V(\mathbf{X}_T, T) = f_j \left(x_T, T \right) = e^{iux_T}$. But the solution is simply the characteristic function for x_T

$$f_j(x_T, T) = \varphi_j \left(u; x_t, v_t, t \right) = E^{\mathbb{Q}_j} \left[e^{iux_T} \right].$$

Consequently, the inversion theorem in Equation (18) can be applied and the probabilities $\mathbb{Q}_j \left(x_T > \ln K \right)$ obtained. Each probability represents the probability of the call option being in-the-money at expiration under the measure $\mathbb{Q}_j$. See the Note on www.FRouah.com for a detailed explanation of this, and for a complete derivation of the Heston model.

References

[1] Heston, S.L. (1993). "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options." *Review of Financial Studies*, Vol. 6, pp 327-343.

[2] Klebaner, F.C. (2005). *Introduction to Stochastic Calculus With Applications, Second Edition.* London, UK: Imperial College Press.