The Feynman-Kac Theorem
by Fabrice Douglas Rouah
www.FRouah.com
www.Volopta.com

In this Note we illustrate the Feynman-Kac theorem in one dimension, and in multiple dimensions. We illustrate the use of the theorem using the Black-Scholes and Heston models. The Feynman-Kac theorem is explained in detail in textbooks such as the one by Klebaner [2].

1 The Theorem in One Dimension

Suppose that \( x_t \) follows the stochastic process
\[
dx_t = \mu(x_t, t) dt + \sigma(x_t, t) dW^Q_t
\] where \( W^Q_t \) is Brownian motion under the measure \( Q \). Let \( V(x_t, t) \) be a differentiable function of \( x_t \) and \( t \) and suppose that \( V(x_t, t) \) follows the partial differential equation (PDE) given by
\[
\frac{\partial V}{\partial t} + \mu(x_t, t) \frac{\partial V}{\partial x} + \frac{1}{2} \sigma(x_t, t)^2 \frac{\partial^2 V}{\partial x^2} - r(x_t, t) V(x_t, t) = 0
\]
and with boundary condition \( V(X_T, T) \). The theorem asserts that \( V(x_t, t) \) has the solution
\[
V(x_t, t) = E^Q \left[ e^{-\int_t^T r(u, u) du} V(X_T, T) \bigg| \mathcal{F}_t \right]. \tag{2}
\]
Note that the expectation is taken under the measure \( Q \) that makes the stochastic term in Equation (1) Brownian motion. The generator of the process in (1) is defined as the operator
\[
\mathcal{A} = \mu(x_t, t) \frac{\partial}{\partial x} + \frac{1}{2} \sigma(x_t, t)^2 \frac{\partial^2}{\partial x^2} \tag{3}
\]
so the PDE in \( V(x_t, t) \) is sometimes written
\[
\frac{\partial V}{\partial t} + \mathcal{A} V(x_t, t) - r(x_t, t) V(x_t, t) = 0. \tag{4}
\]
The Feynman-Kac theorem can be used in both directions. That is,

1. If we know that \( x_t \) follows the process in Equation (1) and we are given a function \( V(x_t, t) \) with boundary condition \( V(X_T, T) \), then we can always obtain the solution for \( V(x_t, t) \) as Equation (2).

2. If we know that the solution to \( V(x_t, t) \) is given by Equation (2) and that \( x_t \) follows the process in (1), then we are assured that \( V(x_t, t) \) satisfies the PDE in Equation (4).
1.1 Example Using the Black-Scholes Model

Let the stock price \( S_t \) be driven by the process

\[
dS_t = \mu S_t dt + \sigma S_t dW_t.
\]

The risk-neutral process is

\[
dS_t = r S_t dt + \sigma S_t dW^Q_t
\]

where \( W^Q_t = \frac{\mu-r}{\sigma} t + W_t \). A derivative written on the stock follows the Black-Scholes PDE

\[
\frac{\partial V}{\partial t} + r S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.
\]

(6)

The generator of the process given by Equation (5) is

\[
A = r S \frac{\partial}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2}
\]

so the Black-Scholes PDE in Equation (6) can also be written exactly as in (4) by substituting \( r(x(u, u)) = r \), a constant, in Equation (4). By the Feynman-Kac Theorem, the time-\( t \) value of the derivative with payoff \( V(S_T, T) \) is the solution in Equation (2)

\[
V(S_t, t) = e^{-r(T-t)} E^Q \left[ V(S_T, T) \mid \mathcal{F}_t \right].
\]

(7)

1.1.1 European Call

The time-\( t \) value \( V(S_t, t) \) of a European call option written on \( S_t \) with strike price \( K \) has the payoff \( V(S_T, T) = \max(S_T - K, 0) \). We substitute this payoff in the Feynman-Kac formula in Equation (7). Hence, the time-\( t \) value of a European call is

\[
V(S_t, t) = e^{-r(T-t)} e^{\Phi(d_1) - \Phi(d_2) K \Phi(d_2)}
\]

where \( d_1 = \frac{\ln S/K + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \), \( d_2 = d_1 - \sigma \sqrt{T-t} \), and \( \Phi(y) \) is the standard normal cdf. See the Note on www.FRouah.com for several ways in which the Black-Scholes European call price (9) can be derived.

1.1.2 Binary Options

The cash-or-nothing call pays a fixed amount \( X \) if \( S_T > K \) and nothing otherwise. We can price this option using the Feynman-Kac theorem under the Black-Scholes dynamics. The boundary condition is \( V(S_T, T) = X 1_{S_T > K} \) so the time-\( t \) value of the cash-or-nothing call is, by Equation (7)

\[
V(S_t, t) = e^{-r(T-t)} E^Q \left[ X 1_{S_T > K} \mid \mathcal{F}_t \right] = X e^{-r(T-t)} Q(S_T > K).
\]
It can be shown that $S_T$ follows the lognormal distribution and that the probability $Q(S_T > K) = \int_K^\infty dF(S_T)$ can be expressed in terms of $d_2$ as $Q(S_T > K) = \Phi(d_2)$. Hence the value of the cash-or-nothing call is

$$V(S_t, t) = Xe^{-r(T-t)}\Phi(d_2).$$

Similarly, the asset-or-nothing call pays $S_T$ if $S_T > K$ and nothing otherwise. The boundary condition is therefore $V(S_T, T) = S_T1_{S_T > K}$ and the time-$t$ value of the asset-or-nothing call is, by Equation (7)

$$V(S_t, t) = e^{-r(T-t)}E_Q[1_{S_T > K} | \mathcal{F}_t] = e^{-r(T-t)}E_Q[S_T > K, \mathcal{F}_t].$$

1.1.3 The European Call Replicated by Binary Options

The binary options in the preceding section show that the European call can be replicated by

1. A long position in an asset-or-nothing call with strike $K$, and
2. A short position in a cash-or-nothing call that pays $K$ and with strike $K$.

This can also be seen by writing the European call formula in Equation (8) as

$$V(S_t, t) = e^{-r(T-t)}E_Q[\max(S_T - K, 0) | \mathcal{F}_t] = e^{-r(T-t)}E_Q[S_T1_{S_T > K} | \mathcal{F}_t] - e^{-r(T-t)}E_Q[K1_{S_T > K} | \mathcal{F}_t].$$

See the Note on www.FRouah.com on the Black-Scholes formula for an explanation of how the lognormal probability $Q(S_T > K)$ and conditional expectation $E_Q[S_T | S_T > K]$ are derived.

2 Multi-Dimensional Version of the Theorem

Suppose that $x_t$ follows the stochastic process in $n$ dimensions

$$dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dW_t^Q$$

where $x_t$ and $\mu(x_t, t)$ are each vectors of dimension $n$, $W_t^Q$ is a vector of dimension $m$ of $Q$-Brownian motion, and $\sigma(x_t, t)$ is a matrix of size $n \times m$. In other
words

\[
\begin{pmatrix}
    \frac{dx_1(t)}{dt} \\
    \vdots \\
    \frac{dx_n(t)}{dt}
\end{pmatrix}
= \begin{pmatrix}
    \mu_1(x_t,t) \\
    \vdots \\
    \mu_n(x_t,t)
\end{pmatrix} dt
+ \begin{pmatrix}
    \sigma_{11}(x_t,t) & \cdots & \sigma_{1m}(x_t,t) \\
    \vdots & \ddots & \vdots \\
    \sigma_{n1}(x_t,t) & \cdots & \sigma_{nm}(x_t,t)
\end{pmatrix} \begin{pmatrix}
    dW_1^Q(t) \\
    \vdots \\
    dW_m^Q(t)
\end{pmatrix}.
\]

The generator of the process is

\[
A = \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\sigma \sigma^T)_{ij} \frac{\partial^2}{\partial x_i \partial x_j}
\] (10)

where for notational convenience \( \mu_i = \mu_i(x_t,t), \sigma = \sigma(x_t,t), \) and \((\sigma \sigma^T)_{ij}\) is element \((i,j)\) of the matrix \(\sigma \sigma^T\) of size \((n \times n)\). The theorem states that the partial differential equation (PDE) in \(V(x_t,t)\) given by

\[
\frac{\partial V}{\partial t} + AV(x_t,t) - r(x_t,t)V(x_t,t) = 0
\] (11)

and with boundary condition \(V(X_T,T)\) has solution

\[
V(x_t,t) = E^Q \left[ e^{-\int_t^T r(x_u,u)du} V(X_T,T) \left| \mathcal{F}_t \right. \right].
\] (12)

### 2.1 Example Using the Heston Model

#### 2.1.1 The Heston Model as a Bivariate Process and its Generator

In Heston’s model [1], Itô’s lemma can be applied to the processes for the stock price \(S_t\) and variance \(v_t\) to produce the processes for the logarithm of the stock price \(x_t = \ln S_t\) and the variance \(v_t\). Under the EMM \(Q\), these are

\[
dx = \left( r - \frac{1}{2} v \right) dt + \sqrt{v} dW_1^Q
\]

\[
dv = \kappa (\theta - v) dt + \sigma \sqrt{v} dW_2^Q.
\] (13)

The process for \(x = (x, v)\) can be written in terms of two independent Brownian motions \(Z_1\) and \(Z_2\) as

\[
d \begin{pmatrix}
    x \\
    v
\end{pmatrix}
= \begin{pmatrix}
    r - \frac{1}{2} v \\
    \kappa (\theta - v)
\end{pmatrix} dt
+ \begin{pmatrix}
    \sqrt{v} \\
    \sigma \sqrt{v} (1 - \rho^2)
\end{pmatrix} \begin{pmatrix}
    dZ_1 \\
    dZ_2
\end{pmatrix},
\] (14)

\[\text{The Brownian motions } W_1 \text{ and } W_2 \text{ have correlation } \rho \text{ and can be expressed in terms of two independent Brownian motions } Z_1 \text{ and } Z_2 \text{ as } dW_1 = dZ_1 \text{ and } dW_2 = \rho dZ_1 + \sqrt{1-\rho^2} dZ_2.\]
To obtain the generator in Equation (10), we need the following matrix from (14)

\[
\begin{pmatrix}
\sigma \rho & 0 \\
\sigma & \sigma^2 (1 - \rho^2)
\end{pmatrix} =
\begin{pmatrix}
1 & \sigma \rho \sqrt{\nu} \\
\sigma \rho \sqrt{\nu} & \sigma \sqrt{\nu (1 - \rho^2)}
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
\sigma \rho \sqrt{\nu} & \sigma \sqrt{\nu (1 - \rho^2)}
\end{pmatrix}
\]

The generator in Equation (10) is therefore

\[
A = \left( r - \frac{1}{2} \right) \frac{\partial}{\partial x} + \kappa (\theta - v) \frac{\partial}{\partial v} + \frac{1}{2} \left[ \nu \frac{\partial^2}{\partial x^2} + \sigma^2 v \frac{\partial^2}{\partial v^2} + 2 \sigma \rho v \frac{\partial^2}{\partial x \partial v} \right].
\]

The PDE in Equation (11) for \( V = V(x, v, t) \) becomes

\[
\frac{\partial V}{\partial t} + \left( r - \frac{1}{2} \right) \frac{\partial V}{\partial x} + \kappa (\theta - v) \frac{\partial V}{\partial v} + \frac{1}{2} \left[ \nu \frac{\partial^2 V}{\partial x^2} + \sigma^2 v \frac{\partial^2 V}{\partial v^2} + 2 \sigma \rho v \frac{\partial^2 V}{\partial x \partial v} - rV \right] = 0
\]

which is Equation (6) of Heston [1] with \( r(x, t) = r \) (a constant), and with \( \lambda(x, v, t) = 0 \).

### 2.1.2 The Call Option Value

In a general setting of non-constant interest rates \( r_u \) the value of a European call option is

\[
C(S_t, t) = \mathbb{E}^Q \left[ e^{-\int_t^T r_u \, du} \max(S_T - K, 0) \right] = \mathbb{E}^Q \left[ B_t \frac{S_T}{B_T} 1_{S_T > K} \right] - K \mathbb{E}^Q \left[ B_t \frac{1}{B_T} S_T 1_{S_T > K} \right],
\]

where \( B_t = \exp \left( \int_0^t r_u \, du \right) \) is time-\( t \) value of the money-market account. Both time-\( t \) expectations \( \mathbb{E}^Q [\cdot] \) are conditional on the time-\( t \) information set \( (x_t, v_t, t) \).

### 2.1.3 Change of Measure

The objective is to end up with expectations in Equation (17) in which only the indicator function \( 1_{S_T > K} \) remains in each. This is done by changing the numeraires in the expectations and using these new numeraires to produce two Radon-Nikodym derivatives. These Radon-Nikodym derivatives will allow the measure \( Q \) to be changed to new measures \( Q_1 \) and \( Q_2 \). That way, the expectations can be expressed as probabilities \( Q_j (S_T > K) \), albeit under different measures. We then express \( Q_j (S_T > K) \) as \( Q_j (x_T > \ln K) \) where \( x_T = \ln S_T \) and
apply the inversion theorem, according to which we can recover the probability from the characteristic function $\varphi_j(u; x_t, v_t, t)$ for $x_T$ as

$$Q_j(x_T > \ln K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{-iu \ln K} \varphi_j(u; x_t, v_t, t)}{iu} \right) du. \quad (18)$$

For the first expectation in the second line of Equation (17), we change the risk-neutral measure from $Q$ to $Q_1$ by using the Radon-Nikodym derivative $Z_t$ defined as

$$Z_t = \frac{dQ}{dQ_1} = \frac{S_t}{S_T},$$

Hence the first expectation in Equation (17) can be written as

$$E^Q \left[ \frac{B_t}{B_T} S_T 1_{S_T > K} \right] = E^{Q_1} \left[ \frac{B_t}{B_T} S_T 1_{S_T > K} Z_t \right] = E^{Q_1} [S_t 1_{S_T > K}] = S_t Q_1 (S_T > K). \quad (19)$$

For the second expectation we use the Radon-Nikodym derivative we use the price of a zero-coupon bond $P_{t,T}$ as the numeraire, where

$$P_{t,T} = E \left[ e^{-\int_t^T r_u du} \right]. \quad (20)$$

We change the risk-neutral measure from $Q$ to $Q_2$ by using the Radon-Nikodym derivative $Y_t$ defined as

$$Y_t = \frac{dQ}{dQ_2} = \frac{P_{t,T}}{B_t/B_T}.$$

Hence the second expectation in Equation (17) can be written as

$$E^Q \left[ \frac{B_t}{B_T} 1_{S_T > K} \right] = E^{Q_2} \left[ \frac{B_t}{B_T} 1_{S_T > K} Y_t \right] = E^{Q_2} [P_{t,T} 1_{S_T > K}] = P_{t,T} Q_2 (S_T > K). \quad (21)$$

Substituting the expectations in Equations (19) and (21) into the valuation formula (17) produces

$$C(S_t, t) = S_t Q_1 (x_T > \ln K) - P_{t,T} K Q_2 (x_T > \ln K)$$

which is the price of a European call in the Heston model.
2.1.4 The Characteristic Function and the Feynman-Kac Theorem

The point of this example is that there is a link between the characteristic functions $\varphi_j$ ($j = 1, 2$) and the Feynman-Kac theorem. In the Heston [1] model, interest rates are constant. From Equation (12), when $r(x, u) = r$, a constant, the value $V(x_t, t)$ becomes

$$V(x_t, t) = e^{-r(T-t)} E^Q [V(X_T, T)].$$

Set $X_T = \ln S_T = x_T$ and consider the functions $f_j(x_T, T) = E^Q_j[e^{iu x_T}]$. By the Feynman-Kac theorem, we know that this is the solution to a function $f(x_t, t)$ that follows the PDE given in Equation (16), written here in terms of $f_j$

$$\frac{\partial f_j}{\partial t} + \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial f_j}{\partial x} + \kappa (\theta - v) \frac{\partial f_j}{\partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 f_j}{\partial x^2} + \sigma \rho v \frac{\partial^2 f_j}{\partial x \partial v} - rf_j = 0$$

and that has boundary condition $V(X_T, T) = f_j(x_T, T) = e^{iu x_T}$. But the solution is simply the characteristic function for $x_T$

$$f_j(x_T, T) = \varphi_j (u; x_t, v_t, t) = E^Q_j[e^{iu x_T}].$$

Consequently, the inversion theorem in Equation (18) can be applied and the probabilities $Q_j(x_T > \ln K)$ obtained. Each probability represents the probability of the call option being in-the-money at expiration under the measure $Q_j$. See the Note on www.FRouah.com for a detailed explanation of this, and for a complete derivation of the Heston model.

References
