T-Forward Measure

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1 Preliminaries

The time-t value $B(t)$ of the money market account is defined as

$$B(t) = e^{\int_0^t r(u)du} \tag{1}$$

where $r(u)$ is the interest rate process. The time-t price $P(t,T)$ of a zero-coupon bond is defined as the expectation

$$P(t,T) = E^{\mathbb{Q}_B}\left[e^{-\int_t^T r(u)du} \middle| \mathcal{F}_t \right]. \tag{2}$$

under an equivalent martingale measure $\mathbb{Q}_B$ that will be defined in Section 2. When interest rates are deterministic, Equation (2) becomes

$$P(t,T) = e^{-\int_t^T r(u)du} = \frac{B(t)}{B(T)}$$

and when interest rates are constant Equation (2) simplifies to

$$P(t,T) = e^{-r(T-t)}.$$

The expectation in (2) is required because the price $P(t,T)$ is established at time t, and depends on the evolution of future interest rates over (t,T). The price of $B(t) = e^{\int_0^t r(u)du}$ is also established at time t, but depends only on interest rates already realized over $(0,t)$ and not on their evolution. Hence, no expectation is required to evaluate $B(t)$. If the instantaneous forward interest rate $f(t,u)$, $u > t$ is known, then $P(t,T)$ can be written without the expected value, so that

$$P(t,T) = e^{-\int_t^T f(t,u)du}.$$

This implies that the instantaneous forward rate is

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}. \tag{3}$$

2 No-Arbitrage Pricing of Derivatives

In the no-arbitrage approach to pricing derivatives, the time-t value $V(t) = V(t,T,S(t))$ of a derivative on an equity price $S(t)$ is obtained by chossing a numeraire $N(t)$ and taking an expectation with respect to an equivalent martingale measure $\mathbb{N}$ under which the discounted value of the derivative a martingale. Hence $V(t)$ is defined from the expression

$$\frac{V(t)}{N(t)} = E^{\mathbb{N}}\left[\left.\frac{V(T)}{N(T)}\right| \mathcal{F}_t\right]. \tag{4}$$

For example, if the numeraire is chosen as the money market account $B(t)$, then Equation (4) can be written

$$\begin{aligned} V(t) &= E^{\mathbb{Q}_B}\left[\left.\frac{B(t)}{B(T)}V(T)\right| \mathcal{F}_t\right] \\ &= E^{\mathbb{Q}_B}\left[\left.e^{-\int_t^T r(u)du}V(T)\right| \mathcal{F}_t\right], \end{aligned} \tag{5}$$

where $\mathbb{Q}_B$ is the equivalent martingale measure that makes $\frac{V(t)}{B(t)}$ a martingale. When interest rates are deterministic, the exponent can be brought out of the expectation, so that Equation (5) becomes

$$V(t) = e^{-\int_t^T r(u)du}E^{\mathbb{Q}_B}\left[\left.V(T)\right| \mathcal{F}_t\right]. \tag{6}$$

Finally, when interest rates are constant ($r(u) = r$) as in the Black-Scholes or Heston models, Equation (6) further simplifies to

$$V(t) = e^{-r(T-t)}E^{\mathbb{Q}_B}\left[\left.V(T)\right| \mathcal{F}_t\right]. \tag{7}$$

2.1 Interest Rate Derivatives

In the case of interest rate derivatives, the value of the derivative is tied to $r(u)$ so $V(t) = V(t, T, r(t))$. Furthermore, interest rates cannot be assumed constant or determinstic, as these assumptions are too restrictive for modeling the underlying process. This implies that the value of an interest rate derivative is given by Equation (4) in the case of a general numeraire, or by Equation (5)

$$V(t, T, r(t)) = E^{\mathbb{Q}_B}\left[\left.e^{-\int_t^T r(u)du}V(T, T, r(T))\right| \mathcal{F}_t\right]. \tag{8}$$

when $B(t)$ is the numeraire. Unfortunately, Equation (8) cannot be simplified to Equation (6) or (7). The expectation in Equation (8) is difficult to evaluate because it involves two terms that each depend on the value of the underlying.

3 T-Forward Measure

We can evaluate the expectation in Equation (8) by using $P(t,T)$ as the numeraire. The equivalent martingale measure associated with using $P(t,T)$ as the numeraire is the *T-forward measure*. The trick that makes this work is the fact that $P(T,T)=1$. Hence, in Equation (4) we use $P(t,T)$ as the numeraire instead of $N(t)$, which produces

$$\frac{V(t)}{P(t,T)} = E^{\mathbb{Q}_T}\left[\left.\frac{V(T)}{P(T,T)}\right|\mathcal{F}_t\right]$$

so that

$$\begin{aligned} V(t) &= P(t,T)\,E^{\mathbb{Q}_T}\left[V(T)|\,\mathcal{F}_t\right] \\ &= E^{\mathbb{Q}_B}\left[\left.e^{-\int_t^T r(u)du}\right|\mathcal{F}_t\right]E^{\mathbb{Q}_T}\left[V(T)|\,\mathcal{F}_t\right] \end{aligned} \tag{9}$$

where $V(t)=V(t,T,r(t))$. Equation (9) is the main result. The expectation under $\mathbb{Q}_B$ of a product of two terms in Equation (8) is converted into the product of two expectations in Equation (9), one under $\mathbb{Q}_B$ and the other under $\mathbb{Q}_T$, which is easier to evaluate. Hence we have the main result of this Note that $V(t)$ can be written as

$$\begin{aligned} V(t) &= E^{\mathbb{Q}_B}\left[\left.e^{-\int_t^T r(u)du}V(T)\right|\mathcal{F}_t\right] \\ &= P(t,T)\,E^{\mathbb{Q}_T}\left[V(T)|\,\mathcal{F}_t\right]. \end{aligned} \tag{10}$$

Bring $P(t,T)$ inside the expectation for $V(T)$ so that Equation (10) can be written

$$\begin{aligned} V(t) &= E^{\mathbb{Q}_B}\left[\left.\frac{B(t)}{B(T)}V(T)\right|\mathcal{F}_t\right] \\ . &= E^{\mathbb{Q}_T}\left[P(t,T)\,V(T)|\,\mathcal{F}_t\right]. \end{aligned}$$

Since $P(T,T)=1$ we can write

$$E^{\mathbb{Q}_B}\left[\left.\frac{B(t)}{B(T)}V(T)\right|\mathcal{F}_t\right] = E^{\mathbb{Q}_T}\left[\left.\frac{P(t,T)}{P(T,T)}V(T)\right|\mathcal{F}_t\right]. \tag{11}$$

3.1 Radon-Nikodym Derivative

The Radon-Nikodym derivative $\frac{d\mathbb{Q}_T}{d\mathbb{Q}_B}$ to change the measure from $\mathbb{Q}_B$ to $\mathbb{Q}_T$ can be obtained by considering the expectations

$$\begin{aligned} E^{\mathbb{Q}_T}\left[\left.\frac{P(t,T)}{P(T,T)}V(T)\right|\mathcal{F}_t\right] &= E^{\mathbb{Q}_B}\left[\left.\frac{P(t,T)}{P(T,T)}V(T)\frac{d\mathbb{Q}_T}{d\mathbb{Q}_B}\right|\mathcal{F}_t\right] \\ &= E^{\mathbb{Q}_B}\left[\left.\frac{B(t)}{B(T)}V(T)\right|\mathcal{F}_t\right] \end{aligned}$$

where the first equality comes from applying the Radon-Nikodym derivative so that $E^{\mathbb{Q}_T}[X]=E^{\mathbb{Q}_B}\left[X\frac{d\mathbb{Q}_T}{d\mathbb{Q}_B}\right]$, and the second equality comes from Equation (11). Hence we have

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}_B} = \frac{B(t)/B(T)}{P(t,T)/P(T,T)} = \frac{e^{-\int_t^T r(u)du}}{P(t,T)}. \tag{12}$$

To summarize, to value an interest rate derivative, it is convenient to select as the numeraire the price of a zero-coupon bond that has the same maturity, T, as the derivative.

4 Forward Rate a Martingale Under $\mathbb{Q}_T$

Using $B(t)$ as the numeraire, the time-t value $P(t,T)$ of the discount bond can be expressed in terms in Equation (4) as

$$\frac{P(t,T)}{B(t)} = E^{\mathbb{Q}_B}\left[\left.\frac{P(T,T)}{B(T)}\right| \mathcal{F}_t\right].$$

Since the discount bond has payoff $P(T,T)=1$, its time-t price is

$$P(t,T) = E^{\mathbb{Q}_B}\left[\left.\frac{B(t)}{B(T)}\times 1\right| \mathcal{F}_t\right] = E^{\mathbb{Q}_B}\left[\left.e^{-\int_t^T r(u)du}\right| \mathcal{F}_t\right].$$

Differentiate with respect to T

$$-\frac{\partial P(t,T)}{\partial T} = E^{\mathbb{Q}_B}\left[\left.e^{-\int_t^T r(u)du} r(T)\right| \mathcal{F}_t\right]$$

Change the numeraire to $P(t,T)$ so that that $E^{\mathbb{Q}_B}[X] = E^{\mathbb{Q}_T}\left[X\frac{d\mathbb{Q}_B}{d\mathbb{Q}_T}\right]$ with $\frac{d\mathbb{Q}_B}{d\mathbb{Q}_T}$ as the reciprocal of Equation (12)

$$\begin{aligned}
-\frac{\partial P(t,T)}{\partial T} &= E^{\mathbb{Q}_T}\left[\left.e^{-\int_t^T r(u)du} r(T)\frac{P(t,T)}{e^{-\int_t^T r(u)du}}\right| \mathcal{F}_t\right] \\
&= P(t,T)E^{\mathbb{Q}_T}\left[\left. r(T)\right| \mathcal{F}_t\right].
\end{aligned}$$

Since, in general, $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$, we have, using Equation (3) that the forward rate is

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T} = E^{\mathbb{Q}_T}\left[\left. r(T)\right| \mathcal{F}_t\right].$$

Since $r(T) = f(T,T)$, this implies that the forward rate $f(t,T)$ is a martingale under the T-forward measure $\mathbb{Q}_T$

$$f(t,T) = E^{\mathbb{Q}_T}\left[\left. f(T,T)\right| \mathcal{F}_t\right].$$