The Three Methods of Pricing Derivatives

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In this Note we illustrate the three methods for pricing derivatives: pricing by no arbitrage, pricing using partial differential equations (PDE), and pricing using the characteristic function (CF). In the first approach a *replicating portfolio* is constructed from tradeable assets, and the replicating portfolio is assumed to be driven by a financing strategy that is *self-financing*. The portfolio replicates the payoff of the derivative at expiry, and because of no arbitrage, it also replicates the value of the derivative at every instant before expiry. We then use the fact that the numeraire that turns the tradeable assets into martingales, also turns the replicating portfolio into a martingale. The value of the derivative is the expected value of the payoff at expiry, discounted by the numeraire.

In the second approach we use a set of risky assets to form a riskless portfolio that mimics the payoff of the derivative. This corresponds to a portfolio whose stochastic component has been removed, so the SDE driving the portfolio dynamics becomes a PDE. We specify a boundary condition corresponding to the payoff of the derivative, which allows us to solve the PDE and obtain the price of the derivative. This PDE can sometimes be solved using the Feynman-Kac theorem. In other cases it must be solved numerically.

In the third approach, the *characteristic function* of the log stock price must be known. In that case, the call price can be expressed using two integrals that each involve the characteristic function, but these integrals must often be approximated with a numerical integration scheme.

We illustrate the equivalence of these three approaches analytically with the Black-Scholes model.

1 Pricing by no Arbitrage

Before explaining this approaches to pricing derivatives, we present some preliminary concepts. In the economy we assume there are N assets with values $S_1(t), \cdots, S_N(t)$ at time t, each of which follows a stochastic process.

1.1 Preliminaries

1.1.1 Self Financing Trading Strategy

A *trading strategy* is a N-dimensional stochastic process $a_1(t), \cdots, a_N(t)$ that represents the allocations into the assets at time t. The time-t value of the of the portfolio is $\Pi(t) = \sum_{i=1}^{N} a_i(t)S_i(t)$. The trading strategy is *self-financing* if the change in the value of the portfolio is due only to changes in the value of the assets and not to inflows or outflows of funds. This implies the strategy is self-financing if

$$d\Pi(t) = d\left(\sum_{i=1}^{N} a_i(t)S_i(t)\right) = \sum_{i=1}^{N} a_i(t)dS_i(t),$$

in other words, if

$$\Pi(t) = \Pi(0) + \sum_{i=1}^{N}\int_0^t a_i(u)dS_i(u).$$

In the case of two assets the portfolio value is $\Pi(t) = a_1(t)S_1(t)+a_2(t)S_2(t)$ and the strategy (a_1, a_2) is self-financing if $d\Pi(t) = a_1(t)dS_1(t) + a_2(t)dS_2(t)$. An *arbitrage opportunity* is a self-financing trading strategy for which $\Pi(t) \leq 0$ and $\Pr[\Pi(T) > 0] = 1$. This implies that the initial value of the portfolio (at time zero) is zero or negative, and the value of the portfolio at time T will be greater than zero with absolute certainty. This means that we start with a portfolio with zero value, or with debt (negative value). At some future time we have positive wealth, and since the strategy is self-financing, no funds are required to produce this wealth. This is a "free lunch."

We also require that the trading strategy be *predictable*. This implies that the allocations $a_i(t)$ are stochastic, but revealed at time t. This is reasonable, since the portfolio is rebalanced at time t so the allocations are known at that time, and at all instants prior. After time t the allocations are again stochastic, up until the next rebalancing time. This implies that when we integrate $d\Pi(t)$ we can write the individual terms as $\int_0^t a_i(u)dS_i(u)$ rather than as $\int_0^t d\left(a_i(u)S_i(u)\right)$.

1.1.2 Derivatives and Replication

The payoff V_T at time T of a derivative is a function of a risky asset. To rule out arbitrage we identify a self-financing trading strategy that produces the same payoff as the derivative, so that $\Pi_T = V_T$. The trading strategy is then a *replicating strategy* and the portfolio is a *replicating portfolio*. If a replicating strategy exists the derivative is *attainable*, and if all derivatives are attainable the economy is complete.

In the absence of arbitrage the trading strategy produces a unique value for the value V_T of the derivative, otherwise an arbitrage opportunity would exist. Not only that, at every time t the value of the derivative, V_t *must* be equal to the value of the replicating strategy, Π_t, so that $\Pi_t = V_t$. Otherwise an arbitrage opportunity exists. Indeed, if $V_t < \Pi_t$ you could buy the derivative, sell the replicating strategy, and lock in an instant profit. At time T both assets would have equal value ($\Pi_T = V_T$) and the value of the bought derivative would cover the sold strategy. If $V_t > \Pi_t$ you could sell the derivative, buy the replicating strategy, and end up with the same outcome at time T. The technique of determining the value of a derivative by using a replicating portfolio is called *pricing by arbitrage* and will be explained in Section 1.2.

1.1.3 Numeraires and EMM

A *numeraire* is an asset with positive price, namely $N_t > 0$ for all t. Any asset with this property can serve as a numeraire. The *relative price* $\tilde{S}_t$ of an asset is its price S_t divided by the numeraire price, so that $\tilde{S}_t = \frac{S_t}{N_t}$ and S is measured in units of N. The probability measures $\mathbb{Q}$ and $\mathbb{P}$ are *equivalent* (written $\mathbb{P} \sim \mathbb{Q}$) if they take on zero values at the same sets ω of the sample space Ω. Hence $\mathbb{P} \sim \mathbb{Q}$ means that

$$\mathbb{P}(\omega) = 0 \Leftrightarrow \mathbb{Q}(\omega) = 0 \quad \text{for all } \omega \in \Omega.$$

A probability measure $\mathbb{Q}$ is an *equivalent martingale measure* (EMM) or a *risk neutral measure* for the numeraire N_t if two conditions are satisfied (1) $\mathbb{Q} \sim \mathbb{P}$, and (2) the relative price $\tilde{S}_t$ is a martingale under $\mathbb{Q}$, that is

$$E^{\mathbb{Q}}\left[\frac{S_T}{N_T} \middle| \mathcal{F}_t \right] = \frac{S_t}{N_t} \quad \text{for} \quad T > t.$$

In a *complete* market, given a numeraire N, we can always find a unique EMM $\mathbb{N}$ such that asset prices discounted by N are martingales under $\mathbb{N}$. Conversely, given an EMM $\mathbb{N}$ we can always find a unique numeraire N such that asset prices discounted by N are martingales under $\mathbb{N}$.

1.1.4 Radon-Nikodym Derivative

To come.

1.1.5 Girsanov's Theorem

To come.

1.1.6 Fundamental Theorem of Arbitrage

The *Fundamental Theorem of Arbitrage* asserts that if the market is complete, then for each numeraire N_t there exists a unique equivalent martingale measure $\mathbb{N}$ such that the relative price of the assets (and consequently, of the replicating portfolio) using that numeraire is a martingale. In other words S_t/N_t is a martingale under $\mathbb{N}$. Hence

$$E^{\mathbb{N}}\left[\frac{S_T}{N_T} \middle| \mathcal{F}_t \right] = \frac{S_t}{N_t}. \tag{1}$$

If we choose another numeraire M_t, then S_t/M_t is no longer a martingale under $\mathbb{N}$, but market completeness assures us that there exists another unique equivalent martingale measure, $\mathbb{M}$, such that S_t/M_t *is* a martingale under $\mathbb{M}$. Hence

$$E^{\mathbb{M}}\left[\frac{S_T}{M_T} \middle| \mathcal{F}_t \right] = \frac{S_t}{M_t}. \tag{2}$$

1.1.7 The Radon-Nikodym Derivative with Numeraires

The Radon-Nikodym derivative used to pass from measure $\mathbb{N}$ to measure $\mathbb{N}$ can be expressed in terms of numeraires N_t and M_t as

$$\mathbb{Z}_t = \frac{d\mathbb{M}}{d\mathbb{N}} = \frac{N_t/N_T}{M_t/M_T}.$$

To see this, combine Equations (1) and (2) to express S_t under each measure as

$$S_t = E^{\mathbb{N}}\left[S_T \frac{N_t}{N_T} \middle| \mathcal{F}_t \right] = E^{\mathbb{M}}\left[S_T \frac{M_t}{M_T} \middle| \mathcal{F}_t \right]. \quad (3)$$

Change the measure on the right hand side of Equation (3)

$$E^{\mathbb{M}}\left[S_T \frac{M_t}{M_T} \middle| \mathcal{F}_t \right] = E^{\mathbb{N}}\left[S_T \frac{M_t}{M_T} \frac{d\mathbb{M}}{d\mathbb{N}} \middle| \mathcal{F}_t \right].$$

Equation (3) becomes

$$E^{\mathbb{N}}\left[S_T \frac{N_t}{N_T} \middle| \mathcal{F}_t \right] = E^{\mathbb{N}}\left[S_T \frac{M_t}{M_T} \frac{d\mathbb{M}}{d\mathbb{N}} \middle| \mathcal{F}_t \right]. \quad (4)$$

Equate the terms inside the expectations of Equation (4), cancel S_T and rearrange terms to obtain

$$\frac{d\mathbb{M}}{d\mathbb{N}} = \frac{N_t/N_T}{M_t/M_T}.$$

Hence, we can take the expectation of $\frac{S_T}{N_T}$ under $\mathbb{N}$, or we can take the expectation of $\frac{S_T}{M_T}$ under $\mathbb{M}$. The Radon-Nikodym derivative allows us to go back and forth between the two EMM as we please.

1.2 Pricing by No Arbitrage

This is also called the martingale approach to pricing. Derivatives are not part of the traded assets $S_1(t), \cdots, S_N(t)$, so cannot be priced directly. Using the arguments in Section 1.1.2, however, we can form a replicating portfolio $\Pi(t) = \sum_{i=1}^{N} a_i(t)S_i(t)$ that replicates the price of the derivative at every time, so that $V_t = \Pi_t$ for every $t > 0$ and $V_T = \Pi_T$. Moreover, the portfolio is traded since each asset is traded. The Fundamental Theorem of Arbitrage in Section 1.1.6 guarantees that given a numeraire N_t, each relative asset will be a martingale under the corresponding measure $\mathbb{N}$, and consequently, so will V_t/N_t since it is a linear combination of martingales The martingale property of V_t/N_t implies that

$$E^{\mathbb{N}}\left[\frac{V_T}{N_T} \middle| \mathcal{F}_t \right] = \frac{V_t}{N_t}, \quad (5)$$

from which the time-t price of the derivative, V_t, is

$$V_t = N_t E^{\mathbb{N}}\left[\frac{V_T}{N_T} \middle| \mathcal{F}_t \right]. \quad (6)$$

We illustrate this with the Black-Scholes formula.

1.2.1 Pricing Black-Scholes by no Arbitrage

In the Black-Scholes economy we have two assets, a stock S_t that follows the SDE in Equation (14) and a deterministic bond B

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ dB_t &= r B_t dt. \end{aligned} \tag{7}$$

We apply Girsanov's theorem so that the process for dS_t becomes

$$dS_t = r S_t dt + \sigma S_t dW_t^{\mathbb{B}} \tag{8}$$

where $dW_t^{\mathbb{B}} = dW_t + \frac{\mu - r}{\sigma} dt$. It is straightforward to solve for $B_t = \exp\left(\int_0^t r du\right) = e^{rt}$. We use B_t as the numeraire so that $\tilde{S}_t = \frac{S_t}{B_t}$ is a martingale under $\mathbb{B}$. The European call has payoff $V_T = (S_T - K)^+$, so in accordance with Equation (6), the time-t price of the call is

$$\begin{aligned} V_t &= B_t E^{\mathbb{B}} \left[\left. \frac{(S_T - K)^+}{B_T} \right| \mathcal{F}_t \right] \\ &= e^{-r(T-t)} E^{\mathbb{B}} \left[(S_T - K)^+ \middle| \mathcal{F}_t \right] \end{aligned} \tag{9}$$

which is identical to Equation (17). As in Section 2.2.1, we can evaluate this expectation directly and obtain the same solution (18) for the European call price. Note that this derivation does not require the Black-Scholes PDE in Equation (16), nor does it require the application of the Feynman-Kac solution in Equation (13). Note also that in this example the replicating portfolio is simply $\Pi_t = S_t$ with $a_1 = 1$.

1.2.2 Changing the Numeraire

The choice of the numeraire, B, is arbitrary, and S can be used instead. In the previous section we saw that $\frac{S_t}{B_t}$ is a martingale under a EMM $\mathbb{B}$. Now we have that $\frac{B_t}{S_t}$ is a martingale, but under a different measure $\mathbb{S}$. In Equation (9) the value V_t of the call is derived from

$$\frac{V_t}{B_t} = E^{\mathbb{B}} \left[\left. \frac{V_T}{B_T} \right| \mathcal{F}_t \right].$$

Equivalently, using S_t as the numeraire, the same value V_t can be derived from

$$\frac{V_t}{S_t} = E^{\mathbb{S}} \left[\left. \frac{V_T}{S_T} \right| \mathcal{F}_t \right]$$

The European call has payoff $V_T = (S_T - K)^+$, so the time-t price of the call is

$$\begin{aligned} V_t &= S_t E^{\mathbb{S}} \left[\left. \frac{(S_T - K)^+}{S_T} \right| \mathcal{F}_t \right] \\ &= S_t E^{\mathbb{S}} \left[\left. \left(1 - \frac{K}{S_T}\right)^+ \right| \mathcal{F}_t \right]. \end{aligned} \tag{10}$$

Even though the expressions in Equation (9) and (10) are different, both produce the same solution Equation (18) for the call price because the expectations are evaluated under different measures. This is illustrated in detail in the Note on www.FRouah.com.

2 Pricing by the PDE

The basic idea of this approach is that we convert a set of stochastic differential equations (SDE) that govern the price dynamics of the assets (stock, derivatives, bonds, etc.) into a partial differential equation (PDE) along with a boundary condition. The PDE is not stochastic and therefore easier to deal with. It is well know that a PDE has infinitely many solutions unless a set of initial values or boundary conditions are specified. In the case of derivatives, this boundary condition is the payoff of the derivative, the functional form of which is known exactly. To price a derivative under this approach, we set up a replicating portfolio that is made up of our risky assets, but which itself is riskless. In some cases, the PDE will have a solution that is available in closed form. In general, however, the PDE will not have a closed-form solution and will need to be solved numerically, working from the boundary condition.

We first present the Feynman-Kac theorem, which is an essential tool for the pricing of derivatives using the PDE approach.

2.1 Feynman-Kac Theorem

Suppose X_t follows the stochastic process

$$dX_t = \mu\left(X_t, t\right) dt + \sigma\left(X_t, t\right) dW_t^{\mathbb{Q}} \tag{11}$$

and that $V = V(X_t, t)$ is a differentiable function of X_t with time-T value $V(X_T, T)$. Suppose that V satisfies the PDE

$$\frac{\partial V}{\partial t} dt + \mu\left(X_t, t\right) \frac{\partial V}{\partial X} + \frac{1}{2} \sigma\left(X_t, t\right)^2 \frac{\partial^2 V}{\partial X^2} - r\left(X_t, t\right) V\left(X_t, t\right) = 0 \tag{12}$$

with boundary condition $V\left(X_T, T\right)$, where $r\left(X_t, t\right)$ is an integrable function of X_t. The theorem stipulates that the solution to V is given by

$$V\left(X_t, t\right) = E^{\mathbb{Q}}\left[e^{-\int_t^T r(X_u, u) du} V\left(S_T, T\right) \middle| \mathcal{F}_t \right]. \tag{13}$$

Note that the expectation is taken under $\mathbb{Q}$, the same measure under which $W^{\mathbb{Q}}$ is Brownian motion. See the Note on www.FRouah.com for examples of how the Feynman-Kac theorem is used in pricing derivatives.

2.2 Black-Scholes PDE

Under the risk-neutral measure $\mathbb{Q}$, the stock price $S = S_t$ is driven by the SDE

$$dS = rSdt + \sigma SdW \tag{14}$$

where W is $\mathbb{Q}$-Brownian motion. Applying Ito's Lemma, the value $V = V(S_t, t)$ of a derivative written on S follows the SDE

$$dV = \left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(\sigma S \frac{\partial V}{\partial S}\right) dW.$$

The idea is to transform the SDE for V into a PDE, and apply a boundary condition, which will allow a solution to the PDE. As in Section 1.1.1 we set up a riskless replicating portfolio Π made up of ϕ units of the derivative and Δ units of the stock, and assume the portfolio is self-financing, so that $d\Pi = \phi dV + \Delta dS$. The portfolio thus follows the SDE

$$\begin{aligned} d\Pi &= \phi dV + \Delta dS \qquad (15) \\ &= \left(\phi\frac{\partial V}{\partial t} + \phi rS\frac{\partial V}{\partial S} + \frac{1}{2}\phi\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \Delta rS\right) dt + \left(\Delta\sigma S + \sigma S\phi\frac{\partial V}{\partial S}\right) dW. \end{aligned}$$

The choices $\phi = 1$ and $\Delta = -\frac{\partial V}{\partial S}$ remove the stochastic component from $d\Pi$ and renders the portfolio riskless. Since the portfolio is riskless, it earns the risk-free rate and the return on the portfolio in small time increment dt is $d\Pi = r\Pi dt$. We substitute for Δ in Equation (15) and equate with $r\Pi dt$, which produces the Black-Scholes PDE for V

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \qquad (16)$$

We are able to set up a riskless portfolio with S and V and end up with a PDE because S and V are each driven by is a single source of uncertainty, namely the Brownian motion W. This makes it possible to remove the stochastic component and end up with a PDE.

The time-t price $V(S_t, t)$ of a European derivative follows the PDE in Equation (16) for a variety of different derivatives. The value of the derivative will be known at its expiry, so that even though $V(S_t, t)$ is a function whose form is unknown, the value of $V(S_T, T)$ is known completely. This implies that we can price different options, depending on the choice of $V(S_T, T)$ that we use as a boundary condition to solve Equation (16).

To solve the Black-Scholes PDE in Equation (16), we note that the Black-Scholes PDE is a special case of the Feynman-Kac PDE in (12). The time-t price of the derivative with payoff $V(S_T, T)$ is given by the solution to the Feynman-Kac equation in (13)

$$V(S_t, t) = e^{-r(T-t)} E\left[\left. V(S_T, T)\right| \mathcal{F}_t\right].$$

This is the object of the following sub-sections.

2.2.1 European Call

The European call with strike K pay $S_T - K$ at expiry if $S_T > K$, and nothing otherwise, so the boundary condition is $V(S_T, T) = (S_T - K)^+$. The time-t

price of the call is given by the solution to the Feynman-Kac equation (13)

$$V(S_t, t) = e^{-r(T-t)} E\left[(S_T - K)^+ \Big| \mathcal{F}_t \right]. \tag{17}$$

We find the distribution for S_T and evaluate the expectation directly. After some manipulations, we obtain the price of the European call

$$V(S_t, t) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2). \tag{18}$$

where

$$d_1 = \frac{\log \frac{S_t}{K} + \left(r + \frac{\sigma^2}{2} \right)(T-t)}{\sigma\sqrt{T-t}} \tag{19}$$

and where $d_2 = d_1 - \sigma\sqrt{T-t}$.

2.2.2 Digital Options

The cash-or-nothing call pays a fixed amount X if $S_T > K$ and nothing otherwise, so the boundary condition is $V(S_T, T) = X\mathbf{1}_{S_T > K}$. The time-$t$ value of the cash-or-nothing call is given by the solution to the Feynman-Kac equation in (13)

$$V(S_t, t) = e^{-r(T-t)} E\left[X\mathbf{1}_{S_T > K} \middle| \mathcal{F}_t \right].$$

Again, we evaluate the expectation directly using the distribution for S_T. This produces

$$V(S_t, t) = X e^{-r(T-t)} \Phi(d_2).$$

Similarly, the asset-or-nothing call pays S_T if $S_T > K$ and nothing otherwise, so the boundary condition is $V(S_T, T) = S_T \mathbf{1}_{S_T > K}$. The time-$t$ value of the asset-or-nothing call is

$$V(S_t, t) = e^{-r(T-t)} E\left[S_T \mathbf{1}_{S_T > K} \middle| \mathcal{F}_t \right]$$

which evaluates to

$$V(S_t, t) = S_t \Phi(d_1).$$

The European call price in Equation (18) can be replicated by a long position in an asset-or-nothing call with strike K, and a short position in a cash-or-nothing call that pays K and with strike K. See the Notes on www.FRouah.com for a complete derivation of the Black-Scholes PDE and of the Black-Scholes call price.

2.3 Black-Scholes PDE With Dividends

This example is taken from Neftci [4]. Suppose the stock price S and the dividend amount D follow the processes

$$\begin{aligned} dS &= \mu S dt + \sigma S dW \\ dD &= \mu_D D dt + \sigma_D D dW^D \end{aligned} \tag{20}$$

where W and W^D are uncorrelated Brownian motions. Now the derivative depends on both S_t and D_t, so we write $V = V(S_t, D_t, t)$. Applying Ito's Lemma to V implies that it follows the differential equation

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial D}dD + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial D^2}(dD)^2 + \frac{\partial^2 V}{\partial S \partial D}dSdD \\ &= \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \mu_D D\frac{\partial V}{\partial D} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial D^2}\sigma_D^2 D^2\right)dt \\ &\quad + \left(\sigma S\frac{\partial V}{\partial S}\right)dW + \left(\sigma_D D\frac{\partial V}{\partial D}\right)dW^D. \end{aligned} \tag{21}$$

Again, form a portfolio of ϕ units of the stock and Δ units of the derivative and assume it is self financing. Hence $d\Pi = \phi dS + \Delta dV$, which is

$$\begin{aligned} d\Pi &= (\phi\mu S)\,dt \\ &\quad +\Delta\left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \mu_D D\frac{\partial V}{\partial D} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial D^2}\sigma_D^2 D^2\right)dt \\ &\quad + \left(\Delta\sigma S\frac{\partial V}{\partial S} + \phi\sigma S\right)dW + \left(\Delta\sigma_D D\frac{\partial V}{\partial D}\right)dW^D. \end{aligned} \tag{22}$$

In an attempt to render the portfolio riskless, in Equation (22) we can select, for example, $\Delta = 1$ and $\phi = -\frac{\partial V}{\partial S}$, which will eliminate the coefficient for dW. The coefficient for dW^D, however, will still remain. Hence, in this example there is no choice of ϕ and Δ that can eliminate all the stochastic components and render the portfolio riskless.

On the other hand, if dD in Equation (20) is dependent on the same source of randomness dW as dS, then it is possible to form a riskless a riskless portfolio since the assets are both subject to a single stochastic process. The portfolio change $d\Pi = \phi dS + \Delta dV$ in Equation (22) becomes

$$\begin{aligned} d\Pi &= (\phi\mu S)\,dt \\ &\quad +\Delta\left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \mu_D D\frac{\partial V}{\partial D} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial D^2}\sigma_D^2 D^2\right)dt \\ &\quad + \left(\Delta\sigma S\frac{\partial V}{\partial S} + \phi\sigma S + \Delta\sigma_D D\frac{\partial V}{\partial D}\right)dW. \end{aligned} \tag{23}$$

The stochastic term in Equation (23) is

$$\Delta\sigma S\frac{\partial V}{\partial S} + \phi\sigma S + \Delta\sigma_D D\frac{\partial V}{\partial D} = \sigma S\left(\Delta\frac{\partial V}{\partial S} + \phi\right) + \Delta\sigma_D D\frac{\partial V}{\partial D}. \tag{24}$$

Let $\Delta = 1$ in Equation (24) so that

$$\sigma S\left(\frac{\partial V}{\partial S} + \phi\right) + \sigma_D D\frac{\partial V}{\partial D} = 0. \tag{25}$$

Solve for ϕ in Equation (25) to obtain

$$\phi = -\frac{\sigma_D D \frac{\partial V}{\partial D} + \sigma S \frac{\partial V}{\partial S}}{\sigma S}.$$

These choices for Δ and ϕ make the portfolio riskless. Hence, when only a single Brownian motion W drives both assets, a portfolio made up of two assets (stock and derivative) in the right proportion can be made riskless. On the other hand, when two Brownian motions W and W^D drive the assets, a portfolio made up of two assets cannot be made riskless. Fortunately, the portfolio *can* be made riskless by using three assets: a stock and two derivatives. This is how the Heston PDE is constructed in the next section.

2.4 Heston PDE

In the model of Heston [3] the stock price $S = S_t$ and the variance $v = v_t$ are each driven by their own stochastic process

$$\begin{aligned} dS &= \mu S dt + \sqrt{v} S dW_1 \\ dv &= \kappa(\theta - v)dt + \sigma\sqrt{v} dW_2 \end{aligned} \tag{26}$$

where W_1 and W_2 are two Brownian motions with correlation ρ.

2.4.1 The Heston Portfolio

In the Black-Scholes model there is a single source of uncertainty, so only a single derivative is needed to form a riskless portfolio. In the Heston model, however, there are two separate sources of uncertainty W_1 and W_2, so we will need two derivatives to set up a riskless portfolio. We form a portfolio consisting of

- Δ units of the stock S,
- one option $V = V(S, v, t)$ that is used to hedge the stock price,
- ϕ units of another option $U = U(S, v, t)$ that is used to hedge the volatility.

Assuming the portfolio is self-financing, the change in portfolio value is $d\Pi = dV + \Delta dS + \phi dU$. Now apply Itō's Lemma to V so that V follows the process

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}dt + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2}dt + \sigma v \rho S\frac{\partial^2 V}{\partial v \partial S}dt.$$

Applying Itō's Lemma to U produces the identical PDE, but in U. We combine the two PDEs for V and U and write the change in portfolio value, $d\Pi$, as

$$\begin{aligned} d\Pi = &\left\{\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma v S\frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2}\right\}dt + \\ &\phi\left\{\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma v S\frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}\right\}dt + \\ &\left\{\frac{\partial V}{\partial S} + \phi\frac{\partial U}{\partial S} + \Delta\right\}dS + \left\{\frac{\partial V}{\partial v} + \phi\frac{\partial U}{\partial v}\right\}dv. \end{aligned} \tag{27}$$

In order for the portfolio to be hedged against movements in the stock and against volatility, the last two terms in Equation (27) involving dS and dv must be zero. This implies that the hedge parameters must be

$$\phi = -\frac{\frac{\partial V}{\partial v}}{\frac{\partial U}{\partial v}} \quad \text{and} \quad \Delta = -\phi\frac{\partial U}{\partial S} - \frac{\partial V}{\partial S}.$$

With these values of ϕ and Δ the portfolio is riskless so that

$$\begin{aligned} d\Pi &= \left\{\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma vS\frac{\partial^2 V}{\partial v\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2}\right\} dt + \\ &\quad \phi\left\{\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial v\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}\right\} dt \end{aligned}$$

which we write as $d\Pi = (A + \phi B)\, dt$. The portfolio earns the risk-free rate, so as before $d\Pi = r\Pi dt$ so that $(A + \phi B)\, dt = r\,(V + \Delta S + \phi U)\, dt$. Dropping dt, substituting for ϕ and re-arranging, produces the equality

$$\frac{A - rV + rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial v}} = \frac{B - rU + rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}}. \tag{28}$$

2.4.2 The Heston PDE for the Stock Price

The left-hand side of Equation (28) is a function of V only, and the right-hand side is a function of U only. This implies that both sides can be written as a function $f(S, v, t)$ of S, v, and t. Following Heston, specify this function as $f(S, v, t) = -\kappa(\theta - v) + \lambda(S, v, t)$, where $\lambda(S, v, t)$ is the price of volatility risk. Write the left-hand side of Equation (28) as $-\kappa(\theta-v)+\lambda(S, v, t)$, substitute for B and rearrange to produce the Heston PDE expressed in terms of the price S

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma vS\frac{\partial^2 V}{\partial v\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} \\ &-rV + rS\frac{\partial V}{\partial S} + \left[\kappa(\theta - v) - \lambda(S, v, t)\right]\frac{\partial V}{\partial v} = 0. \end{aligned} \tag{29}$$

This is Equation (6) of Heston.

2.4.3 The Heston PDE for the Log Stock Price

Let $x = \ln S$ and express the PDE in terms of x, t and v instead of S, t, and v. Take derivatives of V with respect to x and plug into the Heston PDE Equation (29). All the S terms cancel and we obtain the Heston PDE in terms of the log price $x = \ln S$

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}v\frac{\partial^2 V}{\partial x^2} + \left(r - \frac{1}{2}v\right)\frac{\partial V}{\partial x} + \rho\sigma v\frac{\partial^2 V}{\partial v\partial x} + \\ &\frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} - rV + \left[\kappa(\theta - v) - \lambda v\right]\frac{\partial V}{\partial v} = 0 \end{aligned} \tag{30}$$

where, as in Heston, we have written the market price of risk to be a linear function of the volatility, so that $\lambda(S, v, t) = \lambda v$.

2.4.4 Solving the Heston PDE

The PDE in terms of the log price, Equation (30), is the one used by Heston to obtain the call price in his model. The time-t European call price with maturity $\tau = T - t$ in the Heston model is a generalization of Equation (18)

$$C(S_t, v_t, t) = S_t P_1 - e^{-r\tau} K P_2. \tag{31}$$

Here P_1 and P_2 each represents the probability of the call expiring in-the-money at expiry, but under different measures. Obtaining these probabilities is done by inversion of their corresponding characteristic functions. Going from the Heston PDE (30) to the call price (31) is complicated and can be done in several ways. See the Note on www.FRouah.com for a complete derivation of the Heston model.

3 Pricing with the Characteristic Function

This method is called *pricing by characteristic functions*, or *pricing by Fourier transforms*. In the preceding sections we showed that the price of a call option $C(S_t, t)$ can be obtained using the no-arbitrage method, or the PDE method. Both lead to the expectation under the risk neutral measure $\mathbb{Q}$

$$\begin{aligned} C(S_t, t) &= e^{-r\tau} E^{\mathbb{Q}}\left[(S_T - K)^+\right] \qquad (32) \\ &= e^{-r\tau} E^{\mathbb{Q}}\left[S_T \mathbf{1}_{S_T > K}\right] - K e^{-r\tau} E^{\mathbb{Q}}\left[\mathbf{1}_{S_T > K}\right]. \end{aligned}$$

A number of authors, starting with Heston [3] and including Bakshi and Madan [1] and others, have shown that the call price can be expressed as

$$C(S_t, t) = S_t P_1 - K e^{-r\tau} P_2 \tag{33}$$

where

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re}\left[\frac{e^{-i\phi \ln K} f_j(\phi)}{i\phi}\right] d\phi \tag{34}$$

for $j = 1, 2$. In Equation (34), the function f_2 is the characteristic function of the log-stock price under the risk neutral measure $\mathbb{Q}$ that makes W_1 and W_2 in Equation (26) Brownian motion. The function f_1 is the characteristic function of the log-stock price under the risk neutral measure that uses the stock price as the numeraire. Bakshi and Madan show that

$$f_1(\phi) = \frac{f_2(\phi - i)}{f_2(-i)}. \tag{35}$$

This implies that we only have to obtain f_2, the characteristic under the risk neutral measure that uses the bond as the numeraire. Hence we have

$$P_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re}\left[\frac{e^{-i\phi \ln K} f(\phi - i)}{i\phi f(-i)}\right] d\phi. \tag{36}$$

and

$$P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[\frac{e^{-i\phi \ln K} f(\phi)}{i\phi} \right] d\phi. \tag{37}$$

We first obtain the characteristic function f, then obtain the in-the-money probabilities by the integration in (34), and obtain the call price from (31).

To prove Equation (35), note that the function $f(\phi)$ is the characteristic function of $x = \ln S_T$, so that

$$f(\phi) = E^{\mathbb{Q}} \left[e^{i\phi x} \right].$$

Comparing Equations (32) and (33), we see that P_2 is the in-the-money probability under $\mathbb{Q}$

$$P_2 = \mathbb{Q}(S_T > K) = \mathbb{Q}(x_T > \ln K) = \int_{\ln K}^\infty q(x)\, dx.$$

where $q(x)$ denotes the probability density function for the logarithm of the terminal stock price. To evaluate P_2 we first need the Gil-Pelaez inversion formula, which expresses the cumulative distribution function $\mathbb{Q}(x_T \leq x)$ in terms of the characteristic function $f(\phi)$ as

$$\mathbb{Q}(x_T \leq x) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-i\phi x} f(\phi)}{i\phi} d\phi. \tag{38}$$

The density $q(x)$ is obtained by differentiation with respect to x

$$q(x) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\phi x} f(\phi)\, d\phi.$$

It is well-known that the real part of the characteristic function is even, and the imaginary part is odd. This important fact implies that when we integrate over the entire real line, the imaginary part of $e^{-i\phi x} f(\phi)$ will cancel out, which must happen anyway since $q(x)$ is real. Hence, we can simply integrate over the real part, and since the real part is even the integral over $(0, \infty)$ will be equal to the integral over $(-\infty, 0)$. This implies that we can write the density as

$$\begin{aligned} q(x) &= \frac{1}{2\pi} \int_{-\infty}^\infty \text{Re} \left[e^{-i\phi x} f(\phi) \right] d\phi \\ &= \frac{1}{\pi} \int_0^\infty \text{Re} \left[e^{-i\phi x} f(\phi) \right] d\phi. \end{aligned}$$

By an identical argument, $\mathbb{Q}(x_T \leq x)$ can be written from (38) as

$$\mathbb{Q}(x_T \leq x) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \text{Re} \left[\frac{e^{-i\phi x} f(\phi)}{i\phi} \right] d\phi \tag{39}$$

so that the in-the-money probability P_2 is its complement, evaluated at $x = \ln K$

$$\begin{aligned} P_2 &= \mathbb{Q}(x_T > \ln K) \\ &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[\frac{e^{-i\phi \ln K} f(\phi)}{i\phi} \right] d\phi \end{aligned} \tag{40}$$

which is identical to Equation (36).

To obtain P_1 we must invoke a change of numeraire. Consider the Radon-Nikodym derivative

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{e^{x_T}}{E^{\mathbb{Q}}\left[e^{x_T}\right]} = \frac{S_T}{S_t e^{r(T-t)}} = \frac{B_t/B_T}{S_t/S_T}$$

This suggests that we should define a new density function $p(x)$ from $q(x)$ via the Radon-Nikodym derivative as

$$p(x) = \frac{e^{x_T}}{E^{\mathbb{Q}}\left[e^{x_T}\right]} q(x). \tag{41}$$

The characteristic function of $p(x)$ is therefore

$$\begin{aligned} g(\phi) &= E^{\mathbb{P}}\left[e^{i\phi x}\right] \\ &= \int_{-\infty}^{\infty} e^{i\phi x} p(x)\, d\phi \\ &= \int_{-\infty}^{\infty} e^{i\phi x} \frac{e^{x} q(x)}{E^{\mathbb{Q}}\left[e^{x_T}\right]} d\phi \\ &= \frac{\int_{-\infty}^{\infty} e^{i\phi x} e^{x} q(x)\, d\phi}{E^{\mathbb{Q}}\left[e^{x_T}\right]}. \end{aligned} \tag{42}$$

Note that $E^{\mathbb{Q}}\left[e^{x_T}\right]$ is a constant and can be taken out of the integral. Note also that since the characteristic function for x_T is $f(\phi) = E^{\mathbb{Q}}\left[e^{i\phi x_T}\right]$, we have $E^{\mathbb{Q}}\left[e^{x_T}\right] = f(-i)$. Finally, the integral in (42) can be written

$$\int_{-\infty}^{\infty} e^{i(\phi-1)x} q(x)\, d\phi$$

which is equal to the characteristic function for x_T evaluated at $\phi - i$. Hence we have that the characteristic function for the density $p(x)$ can be expressed in terms of the characteristic function for $q(x)$ evaluated at the points $-i$ and $\phi - i$ as

$$g(\phi) = \frac{f(\phi - i)}{f(-i)}.$$

Defining $B_t = e^{rt}$ and with $\tau = T - t$ we can write the first term in Equation (32) as

$$\begin{aligned} e^{-r\tau} E^{\mathbb{Q}}\left[S_T \mathbf{1}_{S_T>K}\right] &= S_t E^{\mathbb{Q}}\left[\frac{B_t/B_T}{S_t/S_T} \mathbf{1}_{S_T>K}\right] \\ &= S_t E^{\mathbb{P}}\left[\frac{B_t/B_T}{S_t/S_T} \mathbf{1}_{S_T>K} \frac{d\mathbb{Q}}{d\mathbb{P}}\right] \\ &= e^{x_t} \mathbb{P}\left(x_T > \ln K\right). \end{aligned}$$

All that remains is to show that $\mathbb{P}(x_T > \ln K)$ can be expressed in the form in (36). Apply the inversion theorem to the characteristic function $g(\phi)$ in (42), which produces

$$\begin{aligned} P_1 &= \mathbb{P}(x_T > \ln K) \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln K} f(\phi - i)}{i\phi f(-i)}\right] d\phi \end{aligned} \tag{43}$$

which is Equation (36). What remains to be demonstrated is that $p(x)$ can serve as a PDF, which means we must have $p(x) \geq 0$ for all $x \geq 0$ and that $p(x)$ must integrate to unity. To show the first requirement, note that from Equation (41) that

$$p(x) = \frac{B_t/B_T}{S_t/S_T} q(x) > 0.$$

Now consider the integral of $p(x)$ over $(0, \infty)$

$$\int_0^\infty p(x)\,dx = \frac{\int_0^\infty e^x q(x)\,dx}{E^{\mathbb{Q}}[e^{x_T}]} = \frac{E^{\mathbb{Q}}[e^{x_T}]}{E^{\mathbb{Q}}[e^{x_T}]} = 1.$$

Hence, if follows that $p(x)$ is a PDF.

4 Equivalence of the Approaches

In this section we use the Black-Scholes model to illustrate that the three approaches produce identical results. This is done by showing that the non-arbitrage method produces the Black-Scholes PDE, then by showing that the characteristic functions, by definition, produce the correct P_1 and P_2

4.0.5 Equivalence of No-Arbitrage Method to PDE Method

To show the equivalence of these approaches, it suffices to show that the no-arbitrage method produces the Black-Scholes PDE. The trick is to find the diffusion for the discounted derivative price $\frac{V_t}{N_t}$ and show that $\frac{V_t}{N_t}$ is a martingale only when the Black-Scholes PDE is satisfied. Under the Black-Scholes model, the no-arbitrage price of a derivative V_t is given by Equation (6) with numeraire N_t as the bond with constant interest rates, $B_t = e^{rt}$. Hence Equation (6) becomes, with $N_t = B_t$,

$$V_t = V(S_t, t) = e^{-r(T-t)} E^{\mathbb{Q}}\left[V(S_T, T) \middle| \mathcal{F}_t\right], \tag{44}$$

where $\mathbb{Q}$ is the measure under which the discounted stock price $S_t/B_t = e^{-rt}S_t$ is a martingale. We saw also in Equation (5) that under $\mathbb{Q}$ the discounted derivative price, $Z(S_t, t) = e^{-rt}V(S_t, t)$, is also a martingale, which we write as $Z_t = e^{-rt}V_t$ for convenience. By Ito's lemma, $Z(S_t, t)$ follows the process

$$dZ_t = \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 Z}{\partial S^2}(dS_t)^2 \tag{45}$$

since $(dt)^2 = (dt)(dS) = 0$. Now by the product rule

$$\frac{\partial Z}{\partial t} = -re^{-rt}V_t + e^{-rt}\frac{\partial V}{\partial t}.$$

Substituting into Equation (45) and substituting for dS_t from (7) produces

$$dZ_t = e^{-rt}\left[-rV_t + \frac{\partial V}{\partial t}\right] + \frac{\partial Z}{\partial S}\left[\mu S_t dt + \sigma S_t dW_t\right] + \frac{1}{2}\frac{\partial^2 Z}{\partial S^2}\sigma^2 S^2 dt. \tag{46}$$

Note that $\frac{\partial Z}{\partial S} = e^{-rt}\frac{\partial V}{\partial S}$ and $\frac{\partial^2 Z}{\partial S^2} = e^{-rt}\frac{\partial V}{\partial S}$. Substituting for this in Equation (46) and re-arranging terms produces

$$\begin{aligned} dZ_t &= e^{-rt}\left[-rV_t + \frac{\partial V}{\partial t} + \mu S_t\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right]dt \\ &\quad + e^{-rt}\left[\sigma S_t\frac{\partial V}{\partial S}\right]dW_t. \end{aligned} \tag{47}$$

Now the expectation under which the no-arbitrage price of the derivative is obtained is taken under the EMM, $\mathbb{Q}$, so the process dZ_t must reflect this. Hence, we use the result in Section 1.2 and make a change of measure to $\mathbb{Q}$ in Equation (47) to obtain

$$\begin{aligned} dZ_t &= e^{-rt}\left[-rV_t + \frac{\partial V}{\partial t} + \mu S_t\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right]dt \\ &\quad + e^{-rt}\left[\sigma S_t\frac{\partial V}{\partial S}\right]\left[dW_t^{\mathbb{Q}} - \left(\frac{\mu - r}{\sigma}\right)dt\right]. \end{aligned} \tag{48}$$

since $dW_t^{\mathbb{Q}} = dW_t + \frac{\mu - r}{\sigma}dt$ from the application of Girsanov's Theorem that produced Equation (8). Re-arranging terms in Equation (48) produces

$$\begin{aligned} dZ_t &= e^{-rt}\left[-rV_t + \frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right]dt \\ &\quad + e^{-rt}\left[\sigma S_t\frac{\partial V}{\partial S}\right]dW_t^{\mathbb{Q}}. \end{aligned} \tag{49}$$

Hence, we have that in Equation (49), $Z_t = e^{-rt}V(S_t, t)$, the discounted time-t derivative price, is a martingale under $\mathbb{Q}$ only when the drift in (49) is zero, namely, when

$$-rV_t + \frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = 0. \tag{50}$$

We recognize Equation (50) as the Black-Scholes PDE in (16). Hence, using the Black-Scholes model as an illustration, clearly the discounted price of the derivative can only be a martingale if the Black-Scholes PDE is satisfied. Hence, the value of V_t that solves the PDE in Equation (50) with the boundary condition $V_T = V(S_T, T)$ is the same value V_t that appears in Equation (44).

4.0.6 The Characteristic Function Method for Black-Scholes

Under the risk neutral measure $\mathbb{Q}$ the log stock price $\ln S_T$ is distributed normal with mean $m_2 = \ln S_t + \left(r - \frac{\sigma^2}{2}\right)\tau$ and variance $v^2 = \sigma^2\tau$. The characteristic function of a normal random variable X with mean μ and variance s^2 is known to be

$$\begin{aligned} f(\phi) &= E^{\mathbb{Q}}\left[e^{i\phi X}\right] \\ &= \exp\left(i\phi\mu - \frac{1}{2}\phi^2 s^2\right). \end{aligned} \tag{51}$$

Hence the characteristic function of $\ln S_T$ is

$$f(\phi) = \exp\left(i\phi\left(\ln S_t + \left(r - \frac{\sigma^2}{2}\right)\tau\right) - \frac{1}{2}\phi^2\sigma^2\tau\right). \tag{52}$$

We know that since $\ln S_T$ is normally distributed, we have that

$$\begin{aligned} P_2 &= \mathbb{Q}\left(x_T > \ln K\right) \\ &= 1 - \Phi\left(\frac{\ln K - m_2}{v}\right) \\ &= \Phi\left(\frac{-\ln K + m_2}{v}\right) \\ &= \Phi\left(d_2\right) \end{aligned}$$

where Φ is the standard normal CDF, and where $d_2 = d_1 - \sigma\sqrt{\tau}$ with d_1 defined in Equation (19). If follows from Equation (37) that

$$\begin{aligned} P_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \operatorname{Re}\left[\frac{e^{-i\phi\ln K} f(\phi)}{i\phi}\right] d\phi \\ &= \Phi\left(d_2\right). \end{aligned}$$

From Equation (36)

$$P_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \operatorname{Re}\left[\frac{e^{-i\phi\ln K} f(\phi - i)}{i\phi f(-i)}\right] d\phi.$$

Straightforward calculations lead to

$$\frac{f(\phi - i)}{f(-i)} = \exp\left(i\phi\left(\ln S_t + \left(r + \frac{\sigma^2}{2}\right)\tau\right) - \frac{1}{2}\phi^2\sigma^2\tau\right).$$

We recognize this to be the characteristic function of a normal random variable with mean $m_1 = \ln S_t + \left(r + \frac{\sigma^2}{2}\right)\tau$ and variance $v^2 = \sigma^2\tau$. It follows that

$$\begin{aligned} P_1 &= \mathbb{P}\left(x_T > \ln K\right) \\ &= \Phi\left(\frac{-\ln K + m_1}{v}\right) \\ &= \Phi\left(d_1\right). \end{aligned}$$

Hence the Black-Scholes call price can be written in the usual form

$$C(S_t, t) = S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2)$$

or in the equivalent form

$$\begin{aligned} C(S_t, t) &= S_t \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \ln K} f(\phi - i)}{i\phi f(-i)} \right] d\phi \right) \\ &\quad - K e^{-r\tau} \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi \ln K} f(\phi)}{i\phi} \right] d\phi \right) \end{aligned}$$

where $f(\phi)$ is from Equation (52).

The result that the call price can be written as $C_T(K) = S_t P_1 - e^{-r\tau} K P_2$ with P_1 and P_2 given in Equations (36) and (37) is a generalization of (18) that began with Heston [3]. It represents a general setup that is valid for many other models besides the Black-Scholes model. What is required is that the characteristic function be known.

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