Variance Swap

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In this Note we present a detailed derivation of the fair value of variance that is used in pricing a variance swap. We describe the approach described by Demeterfi *et al.* [2] and others. We also show how a simpler version can be derived, using the forward price as the threshold in the payoff decomposition that is used in the derivation. The variance swap has a payoff equal to

$$N_{\text{var}}\left(\sigma_R^2 - K_{\text{var}}\right) \tag{1}$$

where N_{var} is the notional, σ_R^2 is the realized annual variance of the stock over the life of the swap, and $K_{\text{var}} = E[\sigma_R^2]$ is the delivery (strike) variance. The objective is to find the value of K_{var}.

1 Stock Price SDE

The variance swap starts by assuming a stock price evolution similar to Black-Scholes, but with time-varying volatility parameter σ_t

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t dW_t.$$

Consider $f(S) = \ln S$ and apply Itō's Lemma

$$d\ln S_t = \left(\mu - \frac{1}{2}\sigma_t^2\right)dt + \sigma_t dW_t$$

so that

$$\frac{1}{2}\sigma_t^2 = \frac{dS_t}{S_t} - d\ln S_t. \tag{2}$$

2 The Variance

In equation (2) take the average variance from $t = 0$ to $t = T$

$$\begin{aligned} V_T &= \frac{1}{T}\int_0^T \sigma_t^2 dt = \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \int_0^T d\ln S_t\right] \qquad (3) \\ &= \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_T}{S_0}\right]. \end{aligned}$$

The variance swap rate K_{var} is the fair value of the variance; that is, it is the expected value of the average variance under the risk neutral measure. Hence

$$\begin{aligned} K_{\text{var}} &= E[V_T] = E\left[\frac{1}{T}\int_0^T \sigma_t^2 dt\right] \qquad (4) \\ &= \frac{2}{T}E\left[\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_T}{S_0}\right] = \frac{2}{T}\left\{rT - E\left[\ln\frac{S_T}{S_0}\right]\right\}. \end{aligned}$$

The term $\frac{dS_t}{S_t}$ represents the rate of return of the underlying, so under the risk neutral measure the average expected return over $[0,T]$ is the annual risk free rate r times the time period T, namely rT. Most of the rest of this note will be devoted to finding an expression for $E\left[\ln\frac{S_T}{S_0}\right]$.

3 Log Contract

The log contract has the payoff function

$$f(S_T) = \frac{2}{T}\left(\ln\frac{S_0}{S_T} + \frac{S_T}{S_0} - 1\right). \qquad (5)$$

Note that $f'(S_T) = \frac{2}{T}(\frac{1}{S_0} - \frac{1}{S_T})$ and $f''(S_T) = \frac{2}{T}\left(\frac{1}{S_T^2}\right)$.

4 Payoff Function Decomposition

Any payoff function $f(S_T)$ as a function of the underlying terminal price $S_T > 0$ can be decomposed as follows

$$\begin{aligned} f(S_T) &= f(S_*) + f'(S_*)(S_T - S_*) + \qquad (6) \\ &\int_0^{S_*} f''(K)(K - S_T)^+ dK + \int_{S_*}^{\infty} f''(K)(S_T - K)^+ dK \end{aligned}$$

where $S_* > 0$ is an arbitrary threshold. See the Note on www.FRouah.com for a derivation of equation (6) using three different approaches. Apply equation (6) to the log contract (5) to get

$$\begin{aligned} \frac{2}{T}\left(\ln\frac{S_0}{S_T} + \frac{S_T}{S_0} - 1\right) &= \frac{2}{T}\left(\ln\frac{S_0}{S_*} + \frac{S_*}{S_0} - 1\right) + \frac{2}{T}\left(\frac{1}{S_0} - \frac{1}{S_*}\right)(S_T - S_*) + \\ &\frac{2}{T}\int_0^{S_*} \frac{1}{K^2}(K - S_T)^+ dK + \frac{2}{T}\int_{S_*}^{\infty} \frac{1}{K^2}(S_T - K)^+ dK. \end{aligned}$$

Cancel $\frac{2}{T}$ from both sides and re-arrange the terms to obtain

$$\begin{aligned} -\ln\frac{S_T}{S_*} &= -\frac{S_T - S_*}{S_*} + \int_0^{S_*} \frac{1}{K^2}(K - S_T)^+ dK + \\ & \int_{S_*}^{\infty} \frac{1}{K^2}(S_T - K)^+ dK. \end{aligned} \tag{7}$$

This is equation (28) of Demeterfi *et al.* [2]. Take expectations on both sides of equation (7), bringing the expectations inside the integrals where needed

$$\begin{aligned} -E\left[\ln\frac{S_T}{S_*}\right] &= -\frac{E[S_T] - S_*}{S_*} + \\ & \int_0^{S_*} \frac{1}{K^2} E\left[(K - S_T)^+\right] dK + \int_{S_*}^{\infty} \frac{1}{K^2} E\left[(S_T - K)^+\right] dK \\ &= -\left[\frac{S_0}{S^*}e^{rT} - 1\right] + \\ & e^{rT}\int_0^{S_*} \frac{1}{K^2} P(K) dK + e^{rT}\int_{S_*}^{\infty} \frac{1}{K^2} C(K) dK \end{aligned} \tag{8}$$

where $P(K) = e^{-rT}E\left[(K - S_T)^+\right]$ is the put price, $C(K) = e^{-rT}E\left[(S_T - K)^+\right]$ is the call price, and where $E[S_T] = S_0 e^{rT} = F_T$ is the time-T forward price of the underlying at time zero when the underlying price is S_0. Now write

$$\ln\frac{S_T}{S_0} = \ln\frac{S_T}{S_*} + \ln\frac{S_*}{S_0}$$

which implies that

$$-E\left[\ln\frac{S_T}{S_0}\right] = -\ln\frac{S_*}{S_0} - E\left[\ln\frac{S_T}{S_*}\right]. \tag{9}$$

Substitute equation (8) into (9) to obtain

$$\begin{aligned} -E\left[\ln\frac{S_T}{S_0}\right] &= -\ln\frac{S_*}{S_0} - \left[\frac{S_0}{S_*}e^{rT} - 1\right] + \\ & e^{rT}\int_0^{S_*} \frac{1}{K^2} P(K) dK + e^{rT}\int_{S_*}^{\infty} \frac{1}{K^2} C(K) dK. \end{aligned} \tag{10}$$

5 Fair Value of Variance

Recall equation (4) for the fair value

$$K_{\text{var}} = \frac{2}{T}\left\{rT - E\left[\ln\frac{S_T}{S_0}\right]\right\}. \tag{11}$$

Substitute equation (10) for $-E\left[\ln \frac{S_T}{S_0}\right]$ to obtain the fair value of variance at inception, namely, at time $t = 0$.

$$\begin{aligned} K_{\text{var}} &= \frac{2}{T}\left\{ rT - \left[\frac{S_0}{S_*}e^{rT} - 1\right] - \ln\frac{S_*}{S_0} + \right. \\ &\quad \left. e^{rT}\int_0^{S_*}\frac{1}{K^2}P(K)dK + e^{rT}\int_{S_*}^{\infty}\frac{1}{K^2}C(K)dK \right\} \end{aligned} \tag{12}$$

This is equation (29) of Demeterfi *et al.* [2].

6 Fair Value Using Forward Price

Sometimes K_{var} is written in a simplified form. To see this, let the threshold S_* in equation (12) be defined as the forward price, $S_* = S_0 e^{rT} = F_T$. After some minor algebra, we arrive at

$$K_{\text{var}} = \frac{2}{T}e^{rT}\left\{\int_0^{F_T}\frac{1}{K^2}P(K)dK + \int_{F_T}^{\infty}\frac{1}{K^2}C(K)dK\right\}. \tag{13}$$

7 Mark-to-Market Value of a Variance Swap

In this section we use the notation of Jacquier and Slaoui [6]. At inception of the variance swap, the swap strike is set to the expected value of the future variance, so the swap has value zero. Going forward, however, the value of the swap can become non-zero. To see this, first denote the denote the average expected variance over the time interval (t, T) as

$$K_{\text{var}}^{t,T} = \frac{1}{T-t}E_t\left[\int_t^T \sigma_u^2 du\right]$$

where $E_t[\cdot]$ denotes the expectation at time t. At inception ($t = 0$) we write

$$K_{\text{var}}^{0,T} = \frac{1}{T}E\left[\int_0^T \sigma_u^2 du\right]$$

which is K_{var} that appears in equation (4), (11), (12), and (13), the value at inception of the variance swap strike. The value at time t of the variance swap strike, denoted Π_t, is the time-t expected value

$$\Pi_t = e^{-r(T-t)}E_t\left[\frac{1}{T}\int_0^T \sigma_u^2 du - K_{\text{var}}^{0,T}\right]. \tag{14}$$

At inception this expected value is zero, but at time t it is not necessarily so. We write equation (14) by breaking up the integral, which produces

$$\begin{aligned}\Pi_t &= e^{-r(T-t)}E_t\left[\frac{t}{T}\cdot\frac{1}{t}\int_0^t \sigma_u^2 du + \frac{T-t}{T}\cdot\frac{1}{T-t}\int_t^T \sigma_u^2 du - K_{\text{var}}^{0,T}\right] \quad (15)\\ &= e^{-r(T-t)}\left\{\frac{t}{T}\sigma_{0,t}^2 + \frac{T-t}{T}E_t\left[\frac{1}{T-t}\int_t^T \sigma_u^2 du\right] - K_{\text{var}}^{0,T}\right\}\\ &= e^{-r(T-t)}\left\{\frac{t}{T}\sigma_{0,t}^2 + \frac{T-t}{T}K_{\text{var}}^{t,T} - K_{\text{var}}^{0,T}\right\}\\ &= e^{-r(T-t)}\left\{\frac{t}{T}\left(\sigma_{0,t}^2 - K_{\text{var}}^{0,T}\right) + \frac{T-t}{T}\left(K_{\text{var}}^{t,T} - K_{\text{var}}^{0,T}\right)\right\}\end{aligned}$$

where $\sigma_{0,t}^2 = \frac{1}{t}\int_0^t \sigma_u^2 du$ is the realized variance at time t, which is known. The last equation in (15) for Π_t is one that is often encountered, such as that which appears in Section 8.6 of Flavell [3], for example. It indicates that the time-t mark-to-market value of the variance swap Π_t is a weighted average of two components

1. The term $\sigma_{0,t}^2 - K_{\text{var}}^{0,T}$, which represents the "accrued value" of the variance swap. Indeed, this is the realized variance up to time t minus the contracted strike.

2. The term $K_{\text{var}}^{t,T} - K_{\text{var}}^{0,T}$, which represents the difference in fair strikes calculated at time 0, and calculated at time t.

Hence, at time t, to obtain Π_t, we need to calculate $\int_0^t \sigma_u^2 du$, which involves only variance that has already been realized. We also need to calculate $K_{\text{var}}^{t,T}$. In the same way that equations (12) or (13) are used with options of maturity T to obtain $K_{\text{var}}^{0,T}$, those same equations can be used with options of maturity $(T-t)$ to obtain $K_{\text{var}}^{t,T}$.

8 Constant Vega of a Variance Swap

Exhibit 1 of Demeterfi *et al.* [2] shows that a portfolio of options weighted inversely by the square of their strikes has a vega which becomes independent of the spot price as the number of options increases. This can be demonstrated by setting up a portfolio Π of weighted options C defined as

$$\Pi = \int_0^\infty w(K)C\left(S,K,\sigma\right)dK \quad (16)$$

where $w(K)$ is the weight associated with the options and $C(S,K,\sigma)$ are their Black-Scholes prices. This portfolio is similar to that appearing in equation (13). However, since we are concerned with vanna, which is identical for calls and puts, we do not have to split up the portfolio into calls and puts–either will do. This implies that we can define $C(S, K, \sigma)$ to be either calls or puts, and we don't need to split up the integral in equation (16) into two integrals. To explain their Exhibit 1, Demeterfi *et al.* [2] demonstrate that the vanna of the portfolio, namely $\frac{\partial^2 \Pi}{\partial \sigma \partial S}$, is zero only when the weights are inversely proportional to K^2.

8.1 Vega of the Portfolio

The Black-Scholes vega $\frac{\partial C}{\partial \sigma}$ is identical for vanilla calls and puts so the vega of the portfolio is

$$\mathcal{V}_\Pi = \frac{\partial \Pi}{\partial \sigma} = \int_0^\infty w(K) \frac{\partial C(S, K, \sigma)}{\partial \sigma} dK. \tag{17}$$

The Black-Scholes vega of an individual option is

$$\frac{\partial C}{\partial \sigma} = \frac{\sqrt{T}}{2\sigma\sqrt{2\pi}} S \exp\left(-\frac{1}{2} d_1^2\right)$$

where $d_1 = \frac{1}{\sigma\sqrt{T}}\left[-\ln(x) + \frac{1}{2}\sigma^2 T\right]$ and $x = K/S$. Hence we can write equation (17) as

$$\mathcal{V}_\Pi = \frac{\sqrt{T}}{2\sigma\sqrt{2\pi}} \int_0^\infty w(K) S \exp\left(-\frac{1}{2} d_1^2\right) dK.$$

Change the variable of integration to x. Hence $dx = \frac{1}{S} dK$ so that $dK = S dx$ and we can write

$$\mathcal{V}_\Pi = \frac{\sqrt{T}}{2\sigma\sqrt{2\pi}} \int_0^\infty w(Sx) S^2 \exp\left(-\frac{1}{2} d_1^2\right) dx. \tag{18}$$

In this last equation, d_1 depends on x only. Hence when we differentiate $\mathcal{V}_\Pi$ with respect to S we only need to differentiate the term $w(Sx)S^2$. This derivative is, by the chain rule

$$\frac{\partial}{\partial S}\left[S^2 w(Sx)\right] = 2S \times w(Sx) + S^2 \times \frac{\partial w(Sx)}{\partial S} x. \tag{19}$$

8.2 Vanna of the Portfolio

Substituting equation (19) into (18) and differentiating with respect to S, the sensitivity of the portfolio vega to S is

$$\frac{\partial^2 \Pi}{\partial \sigma \partial S} = \frac{\partial \mathcal{V}_\Pi}{\partial S} = \frac{\sqrt{T}}{2\sigma\sqrt{2\pi}} \int_0^\infty S \left[2w(Sx) + Sx \frac{\partial w(Sx)}{\partial S}\right] \exp\left(-\frac{1}{2} d_1^2\right) dx.$$

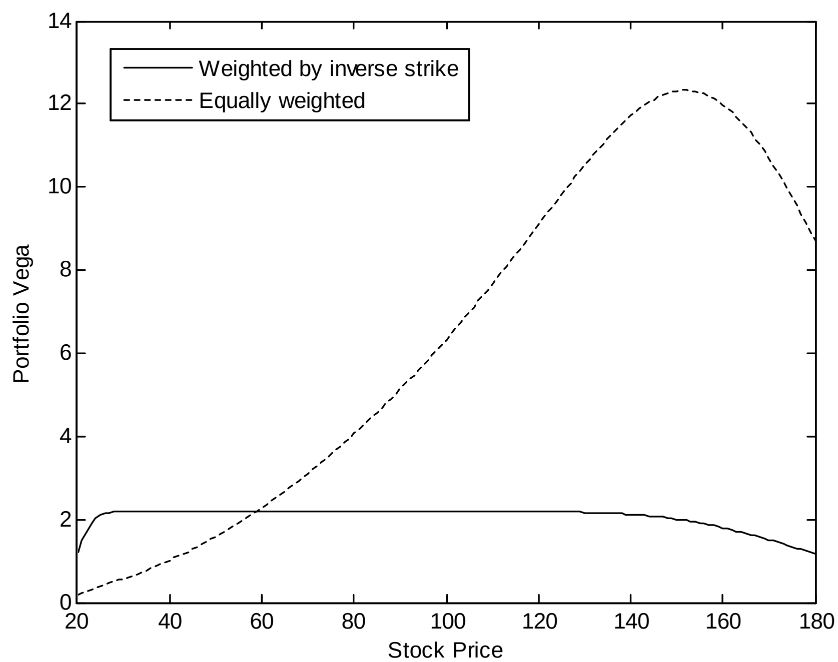
The term inside the square brackets can be written

$$2w(Sx) + Sx \frac{\partial w(Sx)}{\partial S} = 2w(K) + K \frac{\partial w(K)}{\partial K}.$$

Requiring that $\frac{\partial \mathcal{V}_\Pi}{\partial S} = 0$ implies that $2w + K\frac{\partial w}{\partial K} = 0$, or that $w' = -\frac{2w}{K}$. The solution to this differential equation is

$$w(K) \propto \frac{1}{K^2}.$$

Hence, when the weights are chosen to be inversely proportional to K^2, the portfolio vega is insensitive to the spot price so that its vanna is zero. This is illustrated in the following figure, which reproduces part of Exhibit 1 of Demeterfi *et al.* [2]. A portfolio of calls with strikes ranging from \$60 to \$140 in increments of \$10 is formed, and the Black-Scholes vega of the portfolio is calculated by weighing each call equally (dotted line) and by $1/K^2$ (solid line). The interest rate is set to zero, the spot price is set to \$100, the maturity is 6 months and the annual volatility is 20%.



The vega of the equally-weighted portfolio (dotted line) is clearly not constant but increases with the stock price. The vega of the strike-weighted portfolio, on the other hand, is flat in the \$60 to \$140 region, which indicates that its vanna is zero there.

9 Volatility Swap

This is a swap on volatility instead of on variance, so the payoff is

$$N_{\text{vol}}\left(\sigma_R - K_{\text{vol}}\right) \tag{20}$$

where N_{vol} is the notional, σ_R is the realized annual volatility, and K_{vol} is the strike volatility. The values of N_{vol} and K_{vol} can be obtained by writing equation (20) as

$$N_{\text{vol}}\left(\sigma_R - K_{\text{vol}}\right) = \frac{N_{\text{vol}}\left(\sigma_R^2 - K_{\text{vol}}^2\right)}{\sigma_R + K_{\text{vol}}} \approx \frac{N_{\text{vol}}}{2K_{\text{vol}}}\left(\sigma_R^2 - K_{\text{vol}}^2\right). \tag{21}$$

9.1 Naive Estimate of Strike Volatility

Comparing the last term in equation (21) with equation (1), we see that

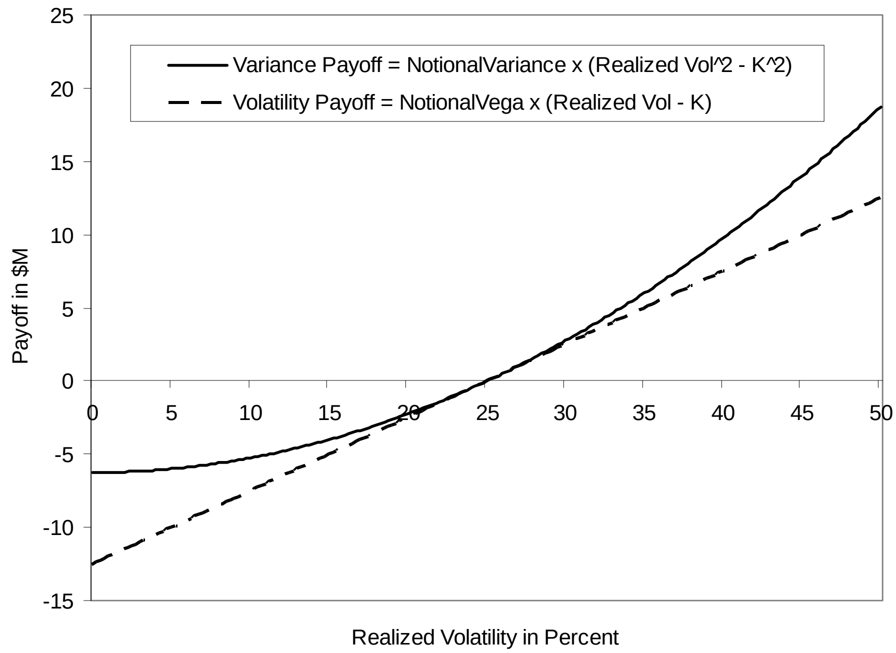
$$\frac{N_{\text{vol}}}{2K_{\text{vol}}} = N_{\text{var}} \quad \text{and} \quad K_{\text{var}} = K_{\text{vol}}^2$$

from which we obtain the naive estimates

$$K_{\text{vol}} = \sqrt{K_{\text{var}}} \quad \text{and} \quad N_{\text{vol}} = 2N_{\text{var}}K_{\text{vol}}.$$

9.2 Vega Notional and Convexity

The payoff of a volatility swap is linear in realized volatility, but the payoff of a variance swap is convex in realized volatility. The notional on volatility, N_{vol} is usually called vega notional. This is because N_{vol} represents the change in the payoff of the swap with a 1 point change in volatility. This is best illustrated with an example. Suppose that the variance notional is $N_{\text{var}} =$ \$10,000, and that the fair estimate of volatilty is 25, so that the variance strike is $K_{\text{var}} = 25^2$. The strike on the volatility is $K_{\text{vol}} = 25$, and the vega notional is $N_{\text{vol}} = 2 \times \$10,000 \times 25 = \$500,000$. In the following figure we plot the payoff from the volatility swap and from the variance swap, as σ_R varies from 0 to 50.



The payoff from the volatility is

$$\begin{aligned}\text{Volatility Payoff} &= N_{\text{vol}} \times (\sigma_R - K_{\text{vol}}) \\ &= \$500,000 \times (\sigma_R - 25)\,,\end{aligned}$$

which is linear in σ_R. When the realized volatility increases by a single point, the payoff increases by exactly $N_{\text{vol}} = \$500,000$. This is represented by the dashed line. The payoff from the variance is

$$\begin{aligned}\text{Variance Payoff} &= N_{\text{var}} \times \left(\sigma_R^2 - K_{\text{var}}\right) \\ &= \$10,000 \times \left(\sigma_R^2 - 25^2\right),\end{aligned}$$

which is convex in σ_R and is represented by the solid line. Suppose that the realized variance experiences a two-point increase from $\sigma_R = 38$ to $\sigma_R = 40$. Then the volatility payoff increases from $\$650,000$ to $\$750,000$ an increase of $\$100,000$ which is exactly $2 \times N_{\text{vol}}$. The variance payoff increases from $\$819,000$ to $\$975,000$ an increase of $\$156,000$.

9.3 Convexity Adjustment

The convexity bias described in [2] is the difference between the last and first term in equation (21). Without loss of generality we can set $N_{\text{vol}} = 1$. Hence

$$\begin{aligned} \text{Convexity Bias} &= \frac{1}{2K_{\text{vol}}}\left(\sigma_R^2 - K_{\text{vol}}^2\right) - (\sigma_R - K_{\text{vol}}) \\ &= \frac{1}{2K_{\text{vol}}}\left(\sigma_R - K_{\text{vol}}\right)^2 . \end{aligned}$$

We approximate the expected value of volatility by the square root of the expected value of variance. Hence we use the approximation $E[\sigma_R] = \sqrt{E[\sigma_R^2]}$. We can find a better approximation to the volatility swap by considering a second order Taylor series expansion of $\sqrt{x}$

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}(x - x_0) + \frac{1}{8\sqrt{x_0^3}}(x - x_0)^2. \tag{22}$$

Set $x = \sigma_R^2 = v$ and $x_0 = E\left[\sigma_R^2\right] = \overline{v}$ in equation (22) to obtain

$$\sqrt{v} \approx \sqrt{\overline{v}} + \frac{1}{2\sqrt{\overline{v}}}(v - \overline{v}) + \frac{1}{8\sqrt{\overline{v}^3}}(v - \overline{v})^2. \tag{23}$$

Take expectations and the middle term on the right hand side of equation (23) drops out to become

$$E\left[\sqrt{v}\right] \approx \sqrt{\overline{v}} + \frac{1}{8\sqrt{\overline{v}^3}} E\left[(v - \overline{v})^2\right].$$

In the original notation

$$E[\sigma_R] \approx \sqrt{E[\sigma_R^2]} + \frac{Var\left[\sigma_R^2\right]}{8E[\sigma_R^2]^{3/2}}.$$

Hence, the loss of accuracy brought on by approximating $E[\sigma_R]$ with $\sqrt{E[\sigma_R^2]}$ can be mitigated by adding the adjustment term $\frac{Var\left[\sigma_R^2\right]}{8E\left[\sigma_R^2\right]^{3/2}}$.

10 Other Issues

10.1 Implementing the Variance Swap Formula

To come. Requires an approximate due to the fact that market prices of puts and calls are not available on a continuum of strikes, but are instead available at discrete strikes, often in increments of \$5 or \$2.50.

10.2 Variance Swap Greeks

To come. Many of the Greeks for variance swaps are available in analytical form.

10.3 Variance Swap as Model-Free Implied Volatility

To come.

10.4 The VIX

To come. The VIX index is a 30-day variance swap on the S&P 500 Index, with convexity adjustment.

11 Variance Swap in Heston's Model

In the Heston (1993) model the stock price S and stock price volatility v each follow their own diffusion, and these diffusions are driven by correlated Brownian motion. Hence

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dZ_t^{(1)} \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dZ_t^{(2)} \end{aligned}$$

with $E\left[dZ_1^{(1)} dZ_2^{(2)}\right] = \rho dt$. The volatility v_t follows a CIR process, and it is straightforward to show that the expected value of v_t, given v_s $(s < t)$ is

$$\begin{aligned} E\left[v_t | v_s\right] &= v_s e^{-\kappa(t-s)} + \theta\left(1 - e^{-\kappa(t-s)}\right) \\ &= \theta + (v_s - \theta) e^{-\kappa(t-s)}. \end{aligned}$$

See, for example, Brigo and Mercurio [1]. As explained by Gatheral [4], a variance swap requires an estimate of the future variance over the $(0, T)$ time period, namely of the total (integrated) variance $w_T = \int_0^T v_t dt$. A fair estimate of w_T is its conditional expectation $E\left[w_T | v_0\right]$. This is given by

$$\begin{aligned} E\left[w_T | v_0\right] &= E\left[\left.\int_0^T v_t dt \right| v_0\right] \\ &= \int_0^T E\left[v_t | v_0\right] dt \\ &= \int_0^T \left[\theta + (v_0 - \theta) e^{-\kappa t}\right] dt \\ &= \theta T + \frac{1 - e^{-\kappa T}}{\kappa}(v_0 - \theta). \end{aligned}$$

Since v_T represents the total variance over $(0, T)$, it must be scaled by T in order to represent a fair estimate of annual variance (assuming that T is expressed in years.) Hence the strike variance for a variance swap is given by

$$\frac{1}{T} E\left[w_T | v_0\right] = \frac{1 - e^{-\kappa T}}{\kappa T}(v_0 - \theta) + \theta.$$

This is the expression on page 138 of Gatheral [4].

References

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